

# The geomagnetic direction problem: The two-dimensional and the three-dimensional axisymmetric case

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## Abstract

We consider the following nonlinear boundary-value problem in the exterior space  $\widehat{V}^d = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| > 1\}$  of a sphere  $S^{d-1}$  in two and three dimensions ( $d = 2, 3$ ): Given a nonvanishing vector field  $\mathbf{D} : S^{d-1} \rightarrow \mathbb{R}^d$ , the so-called direction field, we ask for all harmonic vector fields  $\mathbf{B} : \widehat{V}^d \rightarrow \mathbb{R}^d$  with asymptotic behaviour  $|\mathbf{B}| = O(|\mathbf{x}|^{-\delta})$ ,  $\delta \in \mathbb{N} \setminus \{d-1, d-2\}$  for  $|\mathbf{x}| \rightarrow \infty$ , and which are parallel to  $\mathbf{D}$  on  $S^{d-1}$ , i.e. there is  $a : S^{d-1} \rightarrow \mathbb{R}$  such that  $\mathbf{B} = a\mathbf{D}$ . For  $d = \delta = 3$  this problem is related to the problem of reconstructing the geomagnetic field outside the earth from directional data measured on the earth's surface. The question for uniqueness or non-uniqueness is of particular interest here.

For *fixed* direction field  $\mathbf{D}$  the set of harmonic vector fields  $\mathbf{B}$  form a linear space  $L(\mathbf{D})$ . This space and in particular its dimension is determined in the two-dimensional case and estimated in the three-dimensional axisymmetric case. The first case is much simpler to deal with and serves as a guide for the treatment of the second. Introducing the rotation number  $\varrho$  of a Hölder continuous direction field  $\mathbf{D}$  with respect to  $S^1$  in the case  $d = 2$ ,  $\dim L(\mathbf{D})$  turns out to depend on just two numbers, viz.  $\dim L(\mathbf{D}) = \max\{2(\varrho - \delta) + 1, 0\}$ . Similarly, in the axisymmetric  $d = 3$  case a rotation number  $\varrho$  can be defined along a meridian of  $S^2$ . We obtain then:  $\dim L(\mathbf{D}) \leq \max\{\varrho - \delta + 2, 0\}$ . So, in an axisymmetric setting with  $\delta = 3$  uniqueness is guaranteed only for direction fields with  $\varrho = 2$ .

In the *signed* direction problem only *positive* functions  $a$  are allowed, i.e. the direction including the sign is prescribed on  $S^{d-1}$ . The set of harmonic vector fields  $\mathbf{B}$  is then no longer a linear space but a cone  $C(\mathbf{D})$  in  $L(\mathbf{D})$ . This cone is also determined.

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## 1 Introduction

A well-known problem in geomagnetism is the determination of the magnetic field outside the earth (the determination of “a magnetic field model”) from data measured on the earth's surface. This process is typically plagued by ambiguities arising from the type of data and their availability on the surface. We deal here only with the first source of ambiguity, i.e. we

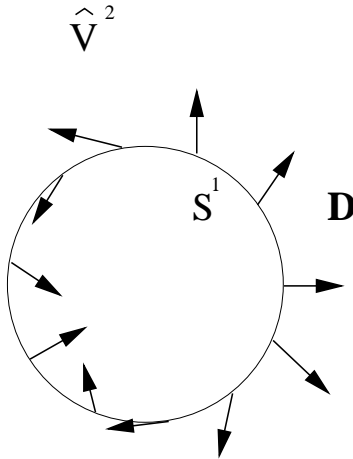


Figure 1: Sketch of the two-dimensional setting

assume the data are given throughout the surface. Concerning the type of data it is well-known that prescription of the normal component of the magnetic field or of the tangential components leads to standard boundary value problems, which can uniquely be solved. This uniqueness is not clear for other types of data like total magnetic intensity or the direction of the magnetic field vector, even when the data are given everywhere on the surface. This paper is concerned with the problem of directional data. Its significance arises in the interpretation of historical magnetic data sets which contain only directional information. Before 1832, when Gauss invented a method of measuring total magnetic intensity, only declination (direction of the horizontal component) and inclination (the angle the field vector makes with the local horizontal) could be measured. The direction problem arises also in the interpretation of palaeomagnetic data records which provide directional information more reliably than information about total intensity (see e.g. Merrill & McElhinny 1983, p.88).

Neglecting deviations from the spherical shape  $S^{d-1}$  of the earth's surface and assuming the exterior region  $\widehat{V}^d$  to be insulating and to be free of sources of magnetic field this problem can be formalized as follows: Given a direction field  $\mathbf{D} \in C(S^{d-1}, \mathbb{R}^d)$  (see Fig. 1) we ask for all nontrivial vector fields  $\mathbf{B} \in C^1(\widehat{V}^d, \mathbb{R}^d) \cap C(\overline{\widehat{V}^d}, \mathbb{R}^d)$  for which a scalar function  $a \in C(S^{d-1}, \mathbb{R})$  exists such that the conditions

$$\left. \begin{aligned} \nabla \times \mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} = 0 & \quad \text{in } \widehat{V}^{d-1}, \\ |\mathbf{B}(\mathbf{x})| = O(|\mathbf{x}|^{-d}) & \quad \text{for } |\mathbf{x}| \rightarrow \infty, \\ \mathbf{B} = a \mathbf{D} & \quad \text{on } S^{d-1} \end{aligned} \right\} \quad (1.1)$$

are satisfied. Some authors refer to problem (1.1) as the “unsigned direction problem” (cf. Hulot et al. 1997). For the “signed direction problem” we require additionally

$$a > 0 \quad \text{on } S^{d-1}. \quad (1.2)$$

In the following we will deal with both types of problems. In fact, we will deal with more general ones, where the decay order  $d$  in (1.1)<sub>2</sub> is replaced by an arbitrary integer  $\delta \in \mathbb{N} \setminus \{d-1, d-2\}$ . This generalization sheds some light upon the uniqueness question without

causing additional trouble. The condition of continuity for the direction field  $\mathbf{D}$  seems to be natural in the formulation of problem (1.1); in fact, continuity of  $\mathbf{D}$  is sufficient to obtain an upper bound on the number of solutions, for existence theorems, however, Hölder continuity turns out to be appropriate.

The standard boundary-value problems of potential theory specify either the normal component or the tangential components (with suitable consistency condition) on the boundary. If combined with an appropriate decay condition at infinity existence and uniqueness of solutions is then guaranteed (cf. e.g. Martensen 1968, p.221f and p.240f). The standard techniques for solving these problems (Hilbert space methods or integral equations) depend crucially on the *linear* relation between solution and boundary data. By contrast problem (1.1) is *non-linear* in this respect: Given two direction fields  $\mathbf{D}_1$  and  $\mathbf{D}_2$  with solutions  $\mathbf{B}_1$  and  $\mathbf{B}_2$ , the superposition  $\mathbf{B}_1 + \mathbf{B}_2$  will in general not be parallel to  $\mathbf{D}_1 + \mathbf{D}_2$  on  $S^{d-1}$ . Not surprisingly existence and uniqueness results comparable to those for the standard problems are not known for problem (1.1) with or without sign condition (1.2) – with one notable exception. The so-called Riemann–Hilbert problem in two dimensions asks for all harmonic vector fields  $\mathbf{B}$  in the unit disk  $D$  with boundary data

$$\mathbf{B} \cdot \mathbf{d} = g \quad \text{on } S^1 \quad (1.3)$$

for given direction field  $\mathbf{d} \neq 0$  and scalar function  $g$  (cf. e.g. Muskhelishvili 1977, p.99). This problem is in fact equivalent to the unsigned problem (1.1) in 2 dimensions with (unphysical) decay order  $\delta = 0$ .<sup>1</sup> This equivalence, however, does neither extend to the signed problem nor to higher dimensions. In any case, since the solution of the Riemann–Hilbert problem is not unique in general, we will not have uniqueness in the problem (1.1) either. Having the above mentioned applications in mind the focus of this paper is on specifying in a suitable way the amount of nonuniqueness of solutions for a given direction field  $\mathbf{D}$ . Of course, due to the homogeneity of problem (1.1), uniqueness is always understood up to a multiplicative constant which remains free.

Several authors have already dealt with problem (1.1) from a geophysical perspective and have, in particular, tried to answer the uniqueness problem. Kono (1976) claimed to have proved uniqueness of the solution of problem (1.1) for  $d = 3$  under an even weaker condition than (1.1)<sub>2</sub>. The proof, however, turned out to be erroneous (Hulot and al. (1997) indicate some loopholes in Kono’s proof) and the claim has been disproved by Proctor & Gubbins (1990) who gave the example of an axisymmetric direction field for which they found numerically three different solutions. On the other side, Hulot et al. (1997) derived in the framework of potential theory an upper bound on the dimension of the solution space  $L(\mathbf{D})$  in terms of the number  $l_{\mathbf{D}}$  of “poles” of the direction field  $\mathbf{D}$  (loci on  $S^2$  with vanishing tangential components):

$$\dim L(\mathbf{D}) \leq l_{\mathbf{D}} - 1. \quad (1.4)$$

In fact, (1.4) is valid in any dimension and is not restricted to a spherical surface.

These findings have been corroborated by Kaiser and Neudert (2004), who determined rigorously the solution spaces of problem (1.1) for certain classes of direction fields in  $d = 2$  and  $d = 3$  with axisymmetry. Based on a Hilbert space formulation of the problem they found in  $d = 3$  for direction fields  $\mathbf{D}_n$ , obtained by restricting exterior axisymmetric  $2^n$ -pole<sup>2</sup> fields on  $S^2$ ,  $\dim L(\mathbf{D}_n) = n$ ,  $n \in \mathbb{N}$  which is just the upper bound in (1.4). For the dipole field

<sup>1</sup>Choose  $\mathbf{d} \perp \mathbf{D}$  and  $g \equiv 0$ , and apply Kelvin’s transformation.

<sup>2</sup>Do not confuse these “poles” with the poles defined above (1.4)!

( $n = 2$ ) this dimension result turned out to be stable with respect to “small” axisymmetric perturbations of  $\mathbf{D}_2$ , i.e.  $\dim L(\mathbf{D}) = 2$  for all axisymmetric direction fields  $\mathbf{D}$  with  $\mathbf{D} - \mathbf{D}_2$  being small in a certain norm. “Small” implies in particular that the rotation number of  $\mathbf{D}$  does not change. Moreover, that paper presented evidence, based on a numerical investigation of combinations of low-order multipole fields, that the rotation number of  $\mathbf{D}$  is in fact decisive for  $\dim L(\mathbf{D})$ . Analogous results have been obtained in  $d = 2$ :  $\dim L(\mathbf{D}_n) = 2(n - 1) + 1$ ,  $n \in \mathbb{N}$ , where  $\mathbf{D}_n$  are the  $d = 2$  analoga of multipole fields, and this result is stable with respect to “small” perturbations for any  $n \in \mathbb{N}$ .

In the present paper problem (1.1) with and without sign condition (1.2) is completely solved in  $d = 2$  and partly solved in  $d = 3$  with axisymmetry for arbitrary (Hölder continuous) direction fields. In  $d = 2$  a complex formulation of our problem is convenient. This leads naturally to the signed problem (1.1), (1.2). The theory of analytic functions allows then the quite explicit construction of solutions (Theorem 2.1). A key ingredient is here the argument principle, which establishes a relation between the rotation number of the direction field, the decay order, and the number of zeroes of the solution in  $\widehat{V}^2$ . The locations of the zeroes are free parameters characterizing the nonuniqueness of the solution. Moreover, a suitable selection of these solutions represent a basis of the solution space of the unsigned problem (1.1) (Theorem 2.2). This approach to the two-dimensional problem has already been proposed by Proctor & Gubbins (1990), however, without working out details or making rigorous statements.

For the treatment of the axisymmetric  $d = 3$  case the  $d = 2$  case serves as a guide. In order to obtain an analogon of the argument principle we use an analytic theory of the degree of mapping (Heinz 1959), which yields again a relation between rotation number, decay order, and number of zeroes. This relation implies an upper bound on the dimension of the solution space of the unsigned problem (Theorem 3.1). To prove existence in the signed problem we use complex notation and make an ansatz analogously to the  $d = 2$  case. Contrary to that case we have now to solve a nonlinear elliptic equation with singular coefficients. In this we succeeded only in part. Reformulating the equation in  $\mathbb{R}^5$  yields an elliptic equation with bounded coefficients but unbounded nonlinearity. Using this formulation and applying fixed point techniques in  $L^p$ -spaces we prove existence of solutions in bounded domains for small data (Theorem 3.2). We show, furthermore, that solvability of this equation in  $\widehat{V}$  without smallness-assumption (not shown here) would provide enough solutions of the (signed) direction problem to make the above upper bound sharp (Theorem 3.3).

## 2 The two-dimensional case

We solve in this section the direction problem in  $d = 2$  for Hölder continuous direction fields with (Theorem 2.1) and without sign condition (Theorem 2.2). Starting point is the complex formulation  $P_{2c}^s$  of the signed problem. Some prerequisites and auxiliary results are formulated as lemmata.

In the complex plane  $\mathbb{C}$  we use the notation  $z = x + iy = r e^{i\varphi}$  with polar coordinates  $(r, \varphi)$ , and for complex functions  $f$  the notation  $f = u + iv$  with real part  $u = \Re f$  and imaginary part  $v = \Im f$ . We associate with the direction field  $\mathbf{D} = D_x \mathbf{e}_x + D_y \mathbf{e}_y$  the complex function  $D_c = D_x + iD_y$ . By introducing the harmonic potential  $U$  for the harmonic vector field  $\mathbf{B} = B_x \mathbf{e}_x + B_y \mathbf{e}_y$  and the analytic function  $F = U + iV$  we have

$$B_x = \partial_x U, \quad B_y = \partial_y U = -\partial_x V.$$

$\mathbf{B}$  is thus associated with the analytic function  $\overline{f} := \overline{F'}$ , where “ $\overline{\cdot}$ ” means complex conjugation. Let now  $\mathbf{D} \neq 0$  be a continuous vector field on  $S^1$ , parametrized with the polar angle  $\varphi$ , i.e.:  $\mathbf{D} \in C([0, 2\pi], \mathbb{R}^2)$ ,  $\varphi(2\pi) = \varphi(0)$ . Denoting the argument of a complex number  $z \neq 0$  with  $\arg(z)$  there is then  $\arg(D_c) \in C([0, 2\pi], \mathbb{R})$ ,

$$\arg(D_c(2\pi)) = 2\pi\rho + \arg(D_c(0)) \quad (2.1)$$

with  $\rho \in \mathbb{Z}$  being the rotation number of  $\mathbf{D}$  with respect to  $S^1$ . We use, finally, the notation  $D = \{z \in \mathbb{C} \mid |z| < 1\}$  and  $\widehat{V} = \widehat{V}^2 = \{z \in \mathbb{C} \mid |z| > 1\}$ . The signed direction problem (1.1), (1.2) takes now the form

**Problem  $P_{2c}^s$ :** Let  $\mathbf{D} = D_x \mathbf{e}_x + D_y \mathbf{e}_y \in C(S^1, \mathbb{R}^2)$ ,  $\mathbf{D} \neq 0$ ,  $D_c = D_x + iD_y$ , and  $\delta \in \mathbb{N} \setminus \{1\}$ . Determine all complex functions  $f \in C(\widehat{V})$ ,  $f \neq 0$  on  $S^1$ , analytic in  $\widehat{V}$ , with

- (i)  $\arg f|_{S^1} = -\arg D_c$ ,
- (ii)  $|f(z)| = O\left(\frac{1}{r^\delta}\right)$  for  $r = |z| \rightarrow \infty$ .

The unsigned problem  $P_{2c}^u$  reads in complex notation:

**Problem  $P_{2c}^u$ :** Let  $\mathbf{D} = D_x \mathbf{e}_x + D_y \mathbf{e}_y \in C(S^1, \mathbb{R}^2)$ ,  $\mathbf{D} \neq 0$ ,  $D_c = D_x + iD_y$ , and  $\delta \in \mathbb{N} \setminus \{1\}$ . Determine all complex functions  $f \in C(\widehat{V})$ , analytic in  $\widehat{V}$ , with

- (i) there is  $a \in C(S^1, \mathbb{R})$  such that  $f|_{S^1} = a \overline{D_c}$ ,
- (ii)  $|f(z)| = O\left(\frac{1}{r^\delta}\right)$  for  $r = |z| \rightarrow \infty$ .

Note that solutions of problem  $P_{2c}^u$  with fixed direction field  $\mathbf{D}$  and decay order  $\delta$  form a linear space  $L_2^\delta(\mathbf{D})$ , and the set of solutions  $C_2^\delta(\mathbf{D})$  of problem  $P_{2c}^s$  is the subset of  $L_2^\delta(\mathbf{D})$  with the additional property  $a > 0$ . This subset is in fact a cone in the space  $L_2^\delta(\mathbf{D})$ , as follows from the representation

$$C_2^\delta(\mathbf{D}) = \left\{ \sum_{n=1}^N \lambda_n f_n \mid f_n \in C_2^\delta(\mathbf{D}), \lambda_n \geq 0, n = 1, \dots, N, N \in \mathbb{N} \right\}.$$

As already noted  $\delta \in \mathbb{N} \setminus \{1\}$  is more general than problem (1.1), which refers to  $\delta = 2$ .

The first lemma formulates the argument principle for analytic functions adapted to our needs (cf. Ahlfors 1979, p.152):

**Lemma 2.1 (argument principle)** Let  $f$  be an analytic function in  $\widehat{V}$  and  $\nu(f, A)$  be the number of zeroes (counted as often as their order indicates) of  $f$  in the annular region  $A$  bounded by the circles  $c_r, c_R$  with radii  $1 < r < R$ . If, furthermore,  $f \neq 0$  on the positively oriented circles, then

$$\nu(f, A) = \frac{1}{2\pi i} \int_{c_R} \frac{f'(z)}{f(z)} dz - \frac{1}{2\pi i} \int_{c_r} \frac{f'(z)}{f(z)} dz. \quad (2.2)$$

Any nontrivial analytic function  $f$  in  $\widehat{V}$  with decay order  $\delta$  has a Laurent expansion

$$f(z) = \sum_{n=\tilde{\delta}}^{\infty} a_n z^{-n}, \quad a_{\tilde{\delta}} \neq 0, \quad \tilde{\delta} \geq \delta \quad (2.3)$$

with “exact” decay order  $\tilde{\delta}$ . Lemma 2.1 yields now a bound on  $\nu(f, \widehat{V})$  in terms of rotation number and decay order  $\tilde{\delta}$  for any solution  $f$  of problem  $P_{2c}^s$ :

**Lemma 2.2** *Let  $f$  be analytic in  $\widehat{V}$ , continuous in  $\overline{\widehat{V}}$ , and  $\neq 0$  on  $\partial\widehat{V} = S^1$ . If  $\varrho$  is the rotation number of  $f$  on  $S^1$  and  $\tilde{\delta}$  the exact decay order, then*

$$\varrho - \tilde{\delta} = \nu(f, \widehat{V}) \geq 0. \quad (2.4)$$

PROOF: The expressions on the right-hand side of (2.2) are just the rotation numbers of  $f$  with respect to  $c_R$  and  $c_r$ , respectively:

$$\frac{1}{2\pi i} \int_{c_r} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{d\varphi} [\arg f(re^{i\varphi})] d\varphi =: \varrho(f, c_r).$$

Since zeroes of an analytic function do not accumulate in  $\widehat{V}$  and  $f \neq 0$  on  $\partial\widehat{V}$  there are no zeroes between  $c_1$  and  $c_{1+\varepsilon}$  for some  $\varepsilon > 0$ . Thus, with Lemma 2.1 we have  $\varrho(f, c_r) = \text{const}$  for  $r \in (1, 1 + \varepsilon)$  and by continuity even for  $r \in [1, 1 + \varepsilon)$  with value

$$\varrho(f, c_r) = \varrho(f, c_1) = -\varrho(D_c, c_1) = -\varrho, \quad r \in [1, 1 + \varepsilon).$$

On the other side, writing the Laurent expansion (2.3) in the form

$$f(z) = a_{\tilde{\delta}} z^{-\tilde{\delta}} \left( 1 + \sum_{n=1}^{\infty} \frac{a_{\tilde{\delta}+n}}{a_{\tilde{\delta}}} z^{-n} \right), \quad a_{\tilde{\delta}} \neq 0$$

shows that  $f \neq 0$  for  $|z| > R_0$  for some  $R_0 > 0$ . Thus,  $\varrho(f, c_R) = \text{const}$  for  $R > R_0$ , and there is a limit

$$\varrho_{\infty}(f) := \lim_{R \rightarrow \infty} \varrho(f, c_R).$$

With

$$\frac{f'(z)}{f(z)} = -\frac{\tilde{\delta}}{z} + \text{analytic function} \quad \text{for } |z| > R_0$$

follows

$$\varrho_{\infty}(f) = -\tilde{\delta}.$$

(2.4) follows now from (2.2) in the limites  $r \rightarrow 1$ ,  $R \rightarrow \infty$ . □

The next lemma formulates the correct decay conditions for a unique solution of Dirichlet’s problem in  $\widehat{V}$  (cf. Leis 1967, p.96f):

**Lemma 2.3** *Let  $\phi \in C(S^1, \mathbb{R})$ . The real boundary-value problem in  $\widehat{V}$ :*

$$\left. \begin{aligned} \Delta q &= 0 && \text{in } \widehat{V}, \\ q &= \phi && \text{on } S^1, \\ q(x, y) &= O(1) && \text{for } r \rightarrow \infty, \\ |\nabla q(x, y)| &= O(r^{-2}) && \text{for } r \rightarrow \infty \end{aligned} \right\} \quad (2.5)$$

*has a unique solution  $q \in C(\overline{\widehat{V}}) \cap C^2(\widehat{V})$ .*

Note that in  $\widehat{V}$  this solution can explicitly be given by Poisson's integral and Kelvin's transformation.

A harmonic function  $q$  in  $\widehat{V}$ , continuous in  $\overline{\widehat{V}}$ , with the decay property (2.5)<sub>4</sub> can be complemented to an analytic function in  $\widehat{V}$ . However, the harmonic conjugate function need not be well-defined on  $\partial\widehat{V}$ , all the less continuous on  $\overline{\widehat{V}}$ . This problem does not arise if  $q$  is Hölder continuous on  $\partial\widehat{V}$ . We have

**Lemma 2.4 (Privaloff)** *Let  $f$  be analytic in the unit disc  $D$ . Let  $f = u + iv$ ,  $u$  be continuous in  $\overline{D}$  and Hölder continuous on  $S^1$ , i.e. there are  $C > 0$  and  $0 < \alpha < 1$  such that*

$$|u(e^{i\varphi}) - u(e^{i\tilde{\varphi}})| \leq C |e^{i\varphi} - e^{i\tilde{\varphi}}|^\alpha, \quad \varphi, \tilde{\varphi} \in [0, 2\pi],$$

then  $f$  is Hölder continuous in  $\overline{D}$ , i.e.

$$|f(z) - f(\tilde{z})| \leq C C_\alpha |z - \tilde{z}|^\alpha, \quad z, \tilde{z} \in \overline{D},$$

with  $C_\alpha > 0$  depending only on  $\alpha$ .

For a proof we refer to (Bers et al. 1964, p.279f). For our needs  $D$  can be replaced in Lemma 2.4 by  $\widehat{V}$  (by the inversion  $z \rightarrow \frac{1}{z}$ ) and, of course,  $u$  can be replaced by the imaginary part  $v$ .

We are now in the position to formulate

**Theorem 2.1** *Let  $\mathbf{D} : [0, 2\pi] \rightarrow \mathbb{R}^2$  be a Hölder continuous direction field (parametrized with the polar angle  $\varphi$ ,  $\mathbf{D}(0) = \mathbf{D}(2\pi)$ ) with rotation number  $\varrho \geq 2$  and  $D_c = D_x + iD_y$ . Let, furthermore,  $\tilde{\delta} \in \mathbb{N} \setminus \{1\}$ ,  $\tilde{\delta} \leq \varrho$  and  $z_1, \dots, z_{\varrho-\tilde{\delta}} \in \widehat{V}$ . Then*

$$f_{z_1 \dots z_{\varrho-\tilde{\delta}}}(z) := e^{g(z)} \prod_{n=1}^{\varrho-\tilde{\delta}} (z - z_n) z^{-\varrho} \quad (2.6)$$

is a solution of the signed problem  $P_{2c}^s$  with exact decay order  $\tilde{\delta}$ . Here,  $g$  is an analytic function in  $\widehat{V}$  with  $\mathcal{I}g$  being the solution of the boundary-value problem (2.5) with boundary function

$$\phi(\varphi) := -\arg D_c(\varphi) - \sum_{n=1}^{\varrho-\tilde{\delta}} \arg(e^{i\varphi} - z_n) + \varrho\varphi. \quad (2.7)$$

Conversely, any solution of  $P_{2c}^s$  with exact decay order  $\tilde{\delta}$  has up to a real multiplicative constant  $m > 0$  the form (2.6) with  $\mathcal{I}g$  determined by (2.5), (2.7) for some numbers  $z_1, \dots, z_{\varrho-\tilde{\delta}} \in \widehat{V}$ .

Let  $S_2^{\tilde{\delta}}(\mathbf{D})$  denote the set of these solutions. The total of solutions of  $P_{2c}^s$  with decay order  $\delta \in \mathbb{N} \setminus \{1\}$  is then

$$C_2^\delta(\mathbf{D}) = \bigcup_{\tilde{\delta} \leq \delta \leq \varrho} S_2^{\tilde{\delta}}(\mathbf{D}). \quad (2.8)$$

PROOF: With

$$\arg(re^{i\varphi} - z_0) \Big|_{\varphi=0}^{\varphi=2\pi} = \begin{cases} 1 & \text{if } |z_0| < r \\ 0 & \text{if } |z_0| > r \end{cases} \quad (2.9)$$

and equation (2.1) follows  $\phi(2\pi) = \phi(0)$ , thus  $\phi(\varphi)$  is an admissible boundary function in (2.5). Let  $q(x, y)$  be the unique harmonic solution of (2.5). Considering the Laurent expansion of the (in  $\widehat{V}$ ) analytic function

$$\partial_y q(x, y) + i \partial_x q(x, y) =: h(z),$$

the residue of  $h$  vanishes, as follows from (2.5)<sub>4</sub>. Thus,  $h$  has an analytic integral function  $g$  in  $\widehat{V}$ ,  $g' = h$ , with imaginary part  $\mathcal{I}g = q$ . The real part  $p$  is determined up to a constant  $p_0$ . Because of Lemma 2.4, it is  $g \in C(\widehat{V})$  and hence  $f_{z_1 \dots z_{\varrho-\tilde{\delta}}} \in C(\widehat{V})$ .

In order to verify condition (i) in  $P_{2c}^s$  consider  $\arg f_{z_1 \dots z_{\varrho-\tilde{\delta}}}$  on  $S^1$ :

$$\begin{aligned} \arg f_{z_1 \dots z_{\varrho-\tilde{\delta}}}(e^{i\varphi}) &= q(\cos \varphi, \sin \varphi) + \sum_{n=1}^{\varrho-\tilde{\delta}} \arg(e^{i\varphi} - z_n) - \varrho \varphi \\ &= \phi(\varphi) + \sum_{n=1}^{\varrho-\tilde{\delta}} \arg(e^{i\varphi} - z_n) - \varrho \varphi \\ &= -\arg D_c(\varphi). \end{aligned}$$

Condition (ii) follows immediately from Lemma 2.2 and  $\nu(f_{z_1 \dots z_{\varrho-\tilde{\delta}}}, \widehat{V}) = \varrho - \tilde{\delta}$ . Therefore,  $f_{z_1 \dots z_{\varrho-\tilde{\delta}}}$  is a solution of problem  $P_{2c}^s$  with exact decay order  $\tilde{\delta}$ .

Let now  $f$  be a solution of  $P_{2c}^s$  with exact decay order  $\tilde{\delta}$ . From Lemma 2.2 follows

$$\nu(f, \widehat{V}) = \varrho - \tilde{\delta}.$$

Let  $z_1, \dots, z_{\varrho-\tilde{\delta}}$  be these zeroes of  $f$  in  $\widehat{V}$ . The function

$$f(z) \prod_{n=1}^{\varrho-\tilde{\delta}} (z - z_n)^{-1} z^\varrho =: eg(z)$$

is analytic and  $\neq 0$  in  $\widehat{V}$ . Defining

$$q(x, y) := \arg eg(z),$$

$q$  is harmonic in  $\widehat{V}$  and  $\in C(\widehat{V})$  (because  $f \in C(\widehat{V})$ ). Moreover, with condition (i) in  $P_{2c}^s$  we have on  $S^1$ :

$$\begin{aligned} q(\cos \varphi, \sin \varphi) &= \arg f(e^{i\varphi}) - \sum_{n=1}^{\varrho-\tilde{\delta}} \arg(e^{i\varphi} - z_n) + \varrho \varphi \\ &= -\arg D_c(\varphi) - \sum_{n=1}^{\varrho-\tilde{\delta}} \arg(e^{i\varphi} - z_n) + \varrho \varphi. \end{aligned}$$

Finally, using condition (ii) and the representation (2.3) for  $f$  we have

$$eg(z) \rightarrow a_{\tilde{\delta}} \neq 0 \quad \text{for } r \rightarrow \infty.$$

This implies the conditions (2.5)<sub>3,4</sub> for the harmonic function  $q(x, y)$ . Thus,  $q$  is a solution of (2.5), (2.7). Complementing  $q$  to the analytic function  $g = p + iq$ ,  $p$  is determined up to the real constant  $p_0$ . Therefore,  $f$  has the representation (2.6) up to the positive factor  $m = e^{p_0}$ .

The last assertion (2.8) in the theorem is trivial. □

As to the unsigned problem  $P_{2c}^u$ , obviously, any function  $f \in C_2^\delta(\mathbf{D})$  is a solution of  $P_{2c}^u$  with decay order  $\delta$  and, more generally, arbitrary linear combinations of functions  $\in C_2^\delta(\mathbf{D})$  solve  $P_{2c}^u$ . Therefore,  $\langle C_2^\delta(\mathbf{D}) \rangle \subset L_2^\delta(\mathbf{D})$ , where  $\langle S \rangle$  denotes the real linear span of the set  $S$ . More precisely we have

**Lemma 2.5** Let  $C_2^\delta(\mathbf{D})$  be the cone of solutions of problem  $P_{2c}^s$  with direction field  $\mathbf{D}$  and decay order  $\delta \in \mathbb{N} \setminus \{1\}$ , and  $L_2^\delta(\mathbf{D})$  the corresponding solution space of the unsigned problem  $P_{2c}^u$ . If  $\varrho \geq \delta$  with  $\varrho$  being the rotation number of  $\mathbf{D}$ , then

$$L_2^\delta(\mathbf{D}) = \langle C_2^\delta(\mathbf{D}) \rangle, \quad (2.10)$$

and

$$\dim L_2^\delta(\mathbf{D}) \geq 2(\varrho - \delta) + 1. \quad (2.11)$$

If  $\varrho < \delta$ , then  $L_2^\delta(\mathbf{D}) = \{0\}$ .

PROOF: Let  $f \in L_2^\delta(\mathbf{D})$  and  $a \in C(S^1, \mathbb{R})$  such that  $f|_{S^1} = a \overline{D_c}$ . In the case  $\varrho \geq \delta$  there is according to Theorem 2.1  $C_2^\delta(\mathbf{D}) \neq \emptyset$ . Choose  $g \in C_2^\delta(\mathbf{D})$  with  $b \in C(S^1, \mathbb{R})$  such that  $g|_{S^1} = b \overline{D_c}$ ,  $b > 0$ . Consider

$$h := f + \lambda g$$

with  $\lambda > 0$  such that  $a + \lambda b > 0$  on  $S^1$ . Obviously, we have  $h \in C_2^\delta(\mathbf{D})$ , and  $f$  has the representation

$$f = h - \lambda g,$$

i.e.  $f \in \langle C_2^\delta(\mathbf{D}) \rangle$ . This proves in (2.10) the nontrivial inclusion  $L_2^\delta(\mathbf{D}) \subset \langle C_2^\delta(\mathbf{D}) \rangle$ .

In the case  $\varrho < \delta$  consider the case  $\varrho \geq 2$  first. According to Theorem 2.1 there is (up to a positive factor) a unique solution  $g \in C_2^\varrho(\mathbf{D}) = S_2^\varrho(\mathbf{D})$ , where  $\varrho$  is the exact decay order of  $g$ . With  $b \in C(S^1, \mathbb{R})$ ,  $b > 0$  and  $\lambda > 0$  as before we have  $f + \lambda g \in C_2^\varrho(\mathbf{D})$ . The uniqueness of  $g \in C_2^\varrho(\mathbf{D})$  implies  $f \equiv 0$ , i.e.  $L_2^\delta(\mathbf{D}) = \{0\}$ .

If  $\varrho < 2$  consider  $\tilde{f}(z) = f(z) \cdot z^{\varrho-2}$  instead of  $f$ . Defining the direction field  $\tilde{\mathbf{D}}$  by  $\tilde{D}_c(\varphi) = D_c(\varphi) \cdot e^{i(2-\varrho)\varphi}$  we find  $\tilde{f} \in L_2^{\delta-\varrho+2}(\tilde{\mathbf{D}})$ .  $\tilde{\mathbf{D}}$  has now rotation number = 2 and we find as above  $\tilde{f} \equiv 0$ , which implies  $f \equiv 0$ .

It remains to prove (2.11). In fact, taking  $f^{(0)} \in S_2^\varrho(\mathbf{D})$ , and arbitrary  $f^{(n)}, \tilde{f}^{(n)} \in S_2^{\varrho-n}(\mathbf{D})$ ,  $n = 1, \dots, \varrho - \delta$  ( $f^{(n)}, \tilde{f}^{(n)}$  “asymptotically independent”) furnishes a system of  $(\varrho - \delta) + 1$  linear independent functions  $\in L_2^\delta(\mathbf{D})$ . To realize this consider the Laurent expansions of  $f^{(0)}, f^{(n)}$ , and  $\tilde{f}^{(n)}$ :

$$\left. \begin{aligned} f^{(0)}(z) &= \sum_{m=\varrho}^{\infty} a_m z^{-m}, & a_\varrho &\neq 0, \\ f^{(n)}(z) &= \sum_{m=\varrho-n}^{\infty} a_m^n z^{-m}, & a_{\varrho-n}^n &\neq 0, \quad n = 1, \dots, \varrho - \delta, \\ \tilde{f}^{(n)}(z) &= \sum_{m=\varrho-n}^{\infty} \tilde{a}_m^n z^{-m}, & \tilde{a}_{\varrho-n}^n &\neq 0, \quad n = 1, \dots, \varrho - \delta. \end{aligned} \right\} \quad (2.12)$$

$f^{(n)}, \tilde{f}^{(n)}$  “asymptotically independent” means,  $f^{(n)}, \tilde{f}^{(n)}$  are chosen such that  $a_{\varrho-n}^n / \tilde{a}_{\varrho-n}^n \notin \mathbb{R}$ . Inserting (2.12) into the linear combination

$$\lambda_0 f^{(0)} + \sum_{n=1}^{\varrho-\delta} (\lambda_n f^{(n)} + \tilde{\lambda}_n \tilde{f}^{(n)}) = 0$$

with real coefficients  $\lambda_0, \lambda_n, \tilde{\lambda}_n$ , and comparing successively the powers  $z^{-\delta}, \dots, z^{-\varrho}$  furnishes

$$\lambda_{\varrho-\delta} = \tilde{\lambda}_{\varrho-\delta} = 0, \quad \dots \quad \lambda_1 = \tilde{\lambda}_1 = 0, \quad \lambda_0 = 0.$$

This proves (2.11). □

The next lemma shows that  $L_2^\delta(\mathbf{D})$  is already generated by the subset  $S_2^\delta(\mathbf{D}) \subset C_2^\delta(\mathbf{D})$ . More precisely, we have

**Lemma 2.6** *Let  $\mathbf{D}$  be a direction field with rotation number  $\varrho$  and  $S_2^{\tilde{\delta}}(\mathbf{D})$  the set of solutions of problem  $P_{2c}^s$  with exact decay order  $\tilde{\delta}$ . If  $2 \leq \delta \leq \tilde{\delta} \leq \varrho$  then*

$$S_2^{\tilde{\delta}}(\mathbf{D}) \subset \langle S_2^\delta(\mathbf{D}) \rangle, \quad (2.13)$$

and

$$\dim \langle S_2^\delta(\mathbf{D}) \rangle = 2(\varrho - \delta) + 1. \quad (2.14)$$

PROOF: We show first  $\dim \langle S_2^\delta(\mathbf{D}) \rangle \geq 2(\varrho - \delta) + 1$ . With the  $2(\varrho - \delta) + 1$  linear independent functions  $f^{(0)}, f^{(n)}, \tilde{f}^{(n)}, n = 1, \dots, \varrho - \delta$  from the proof of the last lemma, we define

$$g^{(0)} := f^{(0)} + f^{(\varrho - \delta)}, \quad g^{(n)} := f^{(n)} + f^{(\varrho - \delta)}, \quad g^{(n + \varrho - \delta)} := \tilde{f}^{(n)} + f^{(\varrho - \delta)}, \quad n = 1, \dots, \varrho - \delta \quad (2.15)$$

These functions are still linear independent, solve problem  $P_{2c}^s$  and have exact decay order  $\delta$ . Therefore,

$$\{g^{(n)} : 0 \leq n \leq 2(\varrho - \delta)\} \subset S_2^\delta(\mathbf{D}),$$

and

$$\dim \langle S_2^\delta(\mathbf{D}) \rangle \geq 2(\varrho - \delta) + 1.$$

Let now  $f \in S_2^{\tilde{\delta}}(\mathbf{D})$  with  $\delta \leq \tilde{\delta} \leq \varrho$ , i.e.  $f$  has a Laurent-expansion

$$f(z) = \sum_{m=\tilde{\delta}}^{\infty} a_m z^{-m}, \quad a_{\tilde{\delta}} \neq 0,$$

and  $\varrho - \tilde{\delta}$  zeroes, say  $z_1, \dots, z_{\varrho - \tilde{\delta}}$ . If  $f$  has a representation

$$f = \sum_{n=0}^{2(\varrho - \delta)} \lambda_n g^{(n)} \quad (2.16)$$

with  $g^{(n)}$  given in (2.15), the real coefficients  $\lambda_n, n = 0, \dots, 2(\varrho - \delta)$  have to satisfy

$$\sum_{n=0}^{2(\varrho - \delta)} \lambda_n g^{(n)}(z_m) = f(z_m) = 0, \quad 1 \leq m \leq \varrho - \tilde{\delta}, \quad (2.17)$$

$$\sum_{n=0}^{2(\varrho - \delta)} \lambda_n b_m^n = a_m = 0, \quad \delta \leq m \leq \tilde{\delta} - 1, \quad (2.18)$$

where  $\{b_m^n : m \geq \delta\}$  are the Laurent-coefficients of  $g^{(n)}$ . After separation in real and imaginary part (2.17), (2.18) constitute a system of  $2(\varrho - \delta)$  real homogeneous linear equations for  $\lambda_n, n = 0, \dots, 2(\varrho - \delta)$ . Let  $\{\mu_n : 0 \leq n \leq 2(\varrho - \delta)\}$  be a nontrivial solution of this system and define

$$g := \sum_{n=0}^{2(\varrho - \delta)} \mu_n g^{(n)}.$$

If  $g \neq 0$  on  $S^1$ , then  $g$  or  $-g \in S_2^{\tilde{\delta}}(\mathbf{D})$  with zeroes  $z_1, \dots, z_{\varrho - \tilde{\delta}}$ . According to Theorem 2.1  $g = cf$  with  $c \neq 0$  hence  $f \in \langle S_2^{\tilde{\delta}}(\mathbf{D}) \rangle$ . If  $g$  has zeroes on  $S^1$ , define

$$\tilde{g} := g + \lambda f$$

with  $\lambda > 0$  such that  $\tilde{g} \neq 0$  on  $S^1$ . Then  $\tilde{g} \in S_2^{\tilde{\delta}}(\mathbf{D})$  as well, and we have with  $c > 0$

$$\tilde{g} = cf \quad \Leftrightarrow \quad g = (c - \lambda)f.$$

Since  $g \neq 0$  we have  $c \neq \lambda$  and hence again  $f \in \langle S_2^{\tilde{\delta}}(\mathbf{D}) \rangle$ .

It remains to prove

$$\dim \langle S_2^{\tilde{\delta}}(\mathbf{D}) \rangle \leq 2(\varrho - \tilde{\delta}) + 1.$$

Setting  $\tilde{\delta} = \delta$  we have just proved that any  $f \in S_2^{\delta}(\mathbf{D})$  has a representation (2.16) by  $g^{(0)}, \dots, g^{(2(\varrho - \delta))}$ . Therefore,  $\langle S_2^{\delta}(\mathbf{D}) \rangle \subset \langle \{g^{(n)} : 0 \leq n \leq 2(\varrho - \delta)\} \rangle$  and  $\dim \langle S_2^{\delta}(\mathbf{D}) \rangle \leq 2(\varrho - \delta) + 1$ . □

Combining Lemmata 2.5 and 2.6 we have

**Theorem 2.2** *Let  $\mathbf{D}$  be a direction field with rotation number  $\varrho$ ,  $L_2^{\delta}(\mathbf{D})$  the solution space of problem  $P_{2c}^u$  with decay order  $\delta \in \mathbb{N} \setminus \{1\}$ , and  $S_2^{\delta}(\mathbf{D})$  the set of solutions of  $P_{2c}^s$  with exact decay order  $\delta$ . Then*

$$L_2^{\delta}(\mathbf{D}) = \begin{cases} \langle S_2^{\delta}(\mathbf{D}) \rangle & \text{if } \varrho \geq \delta \\ \{0\} & \text{if } \varrho < \delta \end{cases}$$

and

$$\dim L_2^{\delta}(\mathbf{D}) = \max\{2(\varrho - \delta) + 1, 0\}.$$

REMARKS:

1) The Riemann-Hilbert problem for the unit disk  $D$  reads: Given Hölder continuous functions  $a, b, c : S^1 \rightarrow \mathbb{R}$ ,  $a^2 + b^2 \neq 0$ , find all complex functions  $f = u + iv \in C(\overline{D})$ , analytic in  $D$ , with boundary condition

$$au - bv = c \quad \text{on } S^1.$$

Setting  $a = D_y$ ,  $b = D_x$ ,  $c = 0$  and applying the inversion  $z \rightarrow \frac{1}{z}$ , the Riemann-Hilbert problem is obviously equivalent to problem  $P_{2c}^u$  with decay order  $\delta = 0$ . Formally, Theorem 2.2. does not cover this unphysical case. But, a short examination shows that Theorem 2.2 holds at least for  $\delta \in \mathbb{N}_0$ . Thus, Theorem 2.2 provides an alternative proof of the homogeneous Riemann-Hilbert problem, which has originally been proved by reduction to singular integral equations (cf. Muskhelishvili 1977, p.203f).

2) The restriction of the boundary in the direction problems  $P_{2c}^{s/u}$  to a circle is not essential. In fact,  $\widehat{V}$  can be replaced by any exterior region  $\widehat{W}$  bounded by a simple closed Hölder continuous curve  $C$ . The Riemann mapping theorem provides a conformal mapping  $R : \widehat{V} \rightarrow \widehat{W}$  with topological extension  $\overline{R} : \overline{\widehat{V}} \rightarrow \overline{\widehat{W}}$  (cf. Behnke & Sommer 1972, p.371). This mapping is unique up to automorphisms of the unit disc (cf. Hurwitz & Courant 1925, p.476f) and, by assumption, Hölder continuous on  $\overline{\widehat{V}}$ . It is not hard to see that, given a direction field  $\mathbf{D}$  on  $C$ , any solution  $f$  of problem  $P_{2c}^s$  in  $\overline{\widehat{V}}$  with direction field  $\mathbf{D} \circ \overline{R}$  generates a solution, viz.  $f \circ \overline{R}^{-1}$ , in  $\overline{\widehat{W}}$  with direction field  $\mathbf{D}$ . In particular,  $\mathbf{D}$  and  $\mathbf{D} \circ \overline{R}$  have the same rotation number and  $f$  and  $f \circ \overline{R}^{-1}$  have the same exact decay order. So, Theorems 2.1 and 2.2 remain valid if  $\widehat{V}$  is replaced by  $\widehat{W}$  and  $S^1$  by  $C$ .

### 3 The three-dimensional axisymmetric case

We consider in this section the direction problem in the three-dimensional axisymmetric case. Questions for uniqueness/non-uniqueness and existence require now quite different methods and will be treated in different subsections. We start with various representations of axisymmetric harmonic fields.

In cylindrical coordinates  $(\rho, \theta, \zeta)$  an axisymmetric harmonic field  $\mathbf{B}$  has only two components  $B_\rho$  and  $B_\zeta$  depending on  $\rho$  and  $\zeta$ , and satisfying the system

$$\left. \begin{aligned} \partial_\zeta B_\rho - \partial_\rho B_\zeta &= 0, \\ \partial_\zeta B_\zeta + \frac{1}{\rho} B_\rho + \partial_\rho B_\rho &= 0. \end{aligned} \right\} \quad (3.1)$$

Introducing a harmonic potential  $\chi(\zeta, \rho)$ ,  $\mathbf{B}$  can be represented as

$$\mathbf{B} = \partial_\zeta \chi \mathbf{e}_\zeta + \partial_\rho \chi \mathbf{e}_\rho$$

with  $\chi$  satisfying the differential equation

$$\partial_\zeta^2 \chi + \frac{1}{\rho} \partial_\rho \chi + \partial_\rho^2 \chi = 0. \quad (3.2)$$

So far  $\mathbf{B}$  and  $\chi$  are defined on the half-plane  $H = \{(\zeta, \rho) \in \mathbb{R}^2 : \rho > 0\}$  bounded by the symmetry axis  $\{\rho = 0\}$ . It is convenient to extend the domain of definition to  $\mathbb{R}^2$  by setting

$$\left. \begin{aligned} B_\zeta(\zeta, -\rho) &:= B_\zeta(\zeta, \rho), \\ B_\rho(\zeta, -\rho) &:= -B_\rho(\zeta, \rho) \end{aligned} \right\} \quad (\zeta, \rho) \in H. \quad (3.3)$$

Note that (3.1)<sub>2</sub> implies (if defined)  $B_\rho(\zeta, 0) = 0$ . The potential  $\chi$  is extended by  $\chi(\zeta, -\rho) := \chi(\zeta, \rho)$ .

On  $\mathbb{R}^2$  we use also polar coordinates  $(r, \varphi)$ , related to  $(\zeta, \rho)$  by  $\zeta = r \cos \varphi$ ,  $\rho = r \sin \varphi$ , and use the notation  $\mathbf{B}(r \cos \varphi, r \sin \varphi) =: \tilde{\mathbf{B}}(r, \varphi)$ . Associated with  $\tilde{\mathbf{B}}$  is again a harmonic potential  $\psi(r, \varphi)$ :

$$\tilde{\mathbf{B}} = \partial_r \psi \mathbf{e}_r + \frac{1}{r} \partial_\varphi \psi \mathbf{e}_\varphi$$

with

$$\frac{1}{r} \partial_r^2 (r\psi) + \frac{1}{r^2 \sin \varphi} \partial_\varphi (\sin \varphi \partial_\varphi \psi) = 0. \quad (3.4)$$

A harmonic field in  $\mathbb{R}^3$  vanishing at infinity decays at least as fast as a dipole field, i.e.

$$|\tilde{\mathbf{B}}| = O(1/r^\delta) \quad \text{for } r \rightarrow \infty \quad (3.5)$$

with  $\delta = 3$ . Faster decay ( $\delta > 3$ ) is also possible.

Finally, it is useful again to introduce a complex formulation: With the identifications

$$\Re z := \zeta, \quad \Im z := \rho, \quad \Re f := B_\zeta, \quad \Im f := -B_\rho \quad (3.6)$$

eqs. (3.1) take the form

$$\partial_{\bar{z}} f - \frac{1}{2} \frac{f - \bar{f}}{z - \bar{z}} = 0, \quad (3.7)$$

where  $\partial_{\bar{z}} := \frac{1}{2}(\partial_{\zeta} + i\partial_{\rho})$ . Here  $f : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  is a function of  $z$  and  $\bar{z}$  satisfying the symmetry condition

$$f(z, \bar{z}) = \overline{f(\bar{z}, z)}, \quad (3.8)$$

corresponding to (3.3). Recall that in two dimensions harmonic fields correspond to complex analytic functions  $h$ , i.e.  $\partial_{\bar{z}}h = 0$ , whereas here we have to solve the linear equation (3.7).

In the following  $S^1$  denotes again the unit circle in  $\mathbb{R}^2$ ,  $\widehat{V}$  the exterior region of  $S^1$ , and  $\mathbf{D} = D_{\zeta} \mathbf{e}_{\zeta} + D_{\rho} \mathbf{e}_{\rho} \neq 0$  a direction field on  $S^1$ .  $\mathbf{D}$  is called symmetric if, when parametrized with the polar angle  $\varphi$ ,  $D_{\zeta}(\varphi) = D_{\zeta}(-\varphi)$ ,  $D_{\rho}(\varphi) = -D_{\rho}(-\varphi)$ ,  $\varphi \in [0, 2\pi]$ . The signed and the unsigned problem in the axisymmetric  $d = 3$ -case read now:

**Problem  $P_{3a}^u$ :** *Let  $\mathbf{D} \in C(S^1, \mathbb{R}^2)$  be a symmetric direction field and  $\delta \in \mathbb{N} \setminus \{1, 2\}$ . Determine all solutions  $\mathbf{B} \in C^1(\widehat{V}) \cap C(\overline{\widehat{V}})$  of (3.1) satisfying the symmetry condition (3.3), the decay condition (3.5), and the boundary condition:*

$$\text{There is } a \in C(S^1, \mathbb{R}) \text{ such that } \mathbf{B}|_{S^1} = a \mathbf{D}. \quad (3.9)$$

The signed problem  $P_{3a}^s$  differs from  $P_{3a}^u$  only in that, additionally,  $a > 0$  in (3.9). The construction of solutions in section 3.2 relies on a complex ansatz motivated by the the 2-dimensional construction. So, more useful is the complex formulation of the signed problem:

**Problem  $P_{3ac}^s$ :** *Let  $\mathbf{D} \in C(S^1, \mathbb{R}^2)$  be a symmetric direction field,  $D_c := D_{\zeta} + iD_{\rho}$ , and  $\delta \in \mathbb{N} \setminus \{1, 2\}$ . Determine all solutions  $f \in C^1(\widehat{V}) \cap C(\overline{\widehat{V}})$  of (3.7) satisfying the symmetry condition (3.8), the decay condition (3.5), and the boundary condition:*

$$\arg f|_{S^1} = -\arg D_c. \quad (3.10)$$

Since the construction in section 3.2 works properly only in bounded regions we formulate, finally, a ‘‘bounded’’ version of the signed direction problem:

**Problem  $P_{3ac}^s(A)$ :** *Let  $A$  be the annular region bounded by the circles  $S^1 = c_1$  and  $c_R$ ,  $R > 1$ , and let  $\mathbf{D} \in C(c_1, \mathbb{R}^2)$ ,  $\widehat{\mathbf{D}} \in C(c_R, \mathbb{R}^2)$  be symmetric direction fields. Determine all solutions  $f \in C^1(A) \cap C(\overline{A})$  of (3.7) satisfying the symmetry condition (3.8) and the boundary conditions:*

$$\arg f|_{c_1} = -\arg D_c, \quad \arg f|_{c_R} = -\arg \widehat{D}_c. \quad (3.11)$$

### 3.1 An upper bound on the dimension of the solution space

The key tool to ‘‘count’’ the number of solutions is in two dimensions the argument principle for complex analytic functions relating the rotation number along the boundary of  $\widehat{V}$  with the decay order at infinity and the number of zeroes in between. The natural generalization of the rotation number for continuous functions (not only in two dimensions) is the degree of mapping. An analytic version thereof relates again the degree with the decay order and the set of zeroes. It reads in our situation:

**Lemma 3.1 (degree of mapping)** Let  $\mathbf{B} : \widehat{V} \rightarrow \mathbb{R}^2$  be a vector field of class  $C^1(\widehat{V})$  with Jacobian matrix  $D\mathbf{B}$  and only isolated zeroes  $\{\mathbf{x}_n : n \in \mathbb{N}\} \subset \widehat{V}$ . Let, furthermore,  $\mathcal{C}$  be the cofactor matrix of  $D\mathbf{B}$  and  $G \Subset \widehat{V}$  be a bounded domain with boundary  $\partial G$  of class  $C^1$  and outer normal  $\nu$ .  $G$  is to contain the finitely many zeroes  $\{\mathbf{x}_n : n = 1, \dots, N\}$  and  $\mathbf{B} \neq 0$  on  $\partial G$ . The degree  $\deg(\mathbf{B}, G)$  of  $\mathbf{B}$  in  $G$  is then defined by

$$\deg(\mathbf{B}, G) := \frac{1}{2\pi} \int_{\partial G} (\mathcal{C} \mathbf{B}) \cdot \nu \frac{1}{|\mathbf{B}|^2} ds$$

and it holds

$$\deg(\mathbf{B}, G) = \sum_{n=1}^N \iota(\mathbf{B}, \mathbf{x}_n), \quad (3.12)$$

where  $\iota(\mathbf{B}, \mathbf{x}_n)$  is the index of  $\mathbf{B}$  in  $\mathbf{x}_n$ , defined by

$$\iota(\mathbf{B}, \mathbf{x}_n) := \deg(\mathbf{B}, D(\mathbf{x}_n))$$

with  $D(\mathbf{x}_n)$  being a disc in  $G$  around  $\mathbf{x}_n$  free of further zeroes.

For a proof we refer to (Heinz 1959, in particular, Theorem 4).

Now let  $G$  be the annular region  $A$  bounded by the circles  $c_R$  and  $c_{\widetilde{R}}$  with  $1 < R < \widetilde{R}$ . Evaluating in polar coordinates  $(r, \varphi)$  the degree for this region yields

$$\begin{aligned} \deg(\mathbf{B}, G) &= \frac{1}{2\pi} \int_0^{2\pi} \left( (\widetilde{\mathbf{B}}_\zeta \partial_\varphi \widetilde{\mathbf{B}}_\rho - \widetilde{\mathbf{B}}_\rho \partial_\varphi \widetilde{\mathbf{B}}_\zeta) \frac{1}{|\widetilde{\mathbf{B}}|^2} \right) \Big|_{r=\widetilde{R}} d\varphi \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \left( (\widetilde{\mathbf{B}}_\zeta \partial_\varphi \widetilde{\mathbf{B}}_\rho - \widetilde{\mathbf{B}}_\rho \partial_\varphi \widetilde{\mathbf{B}}_\zeta) \frac{1}{|\widetilde{\mathbf{B}}|^2} \right) \Big|_{r=R} d\varphi, \end{aligned} \quad (3.13)$$

which is the difference of rotation numbers  $\varrho(\mathbf{B}, c_{\widetilde{R}}) - \varrho(\mathbf{B}, c_R)$  of  $\mathbf{B}$  along  $c_{\widetilde{R}}$  and  $c_R$ , respectively. If we switch from components  $(B_\zeta, B_\rho)$  to  $(B_r, B_\varphi)$  we find

$$\int_0^{2\pi} (\widetilde{\mathbf{B}}_\zeta \partial_\varphi \widetilde{\mathbf{B}}_\rho - \widetilde{\mathbf{B}}_\rho \partial_\varphi \widetilde{\mathbf{B}}_\zeta) \frac{1}{|\widetilde{\mathbf{B}}|^2} d\varphi = 2\pi + \int_0^{2\pi} (\widetilde{\mathbf{B}}_r \partial_\varphi \widetilde{\mathbf{B}}_\varphi - \widetilde{\mathbf{B}}_\varphi \partial_\varphi \widetilde{\mathbf{B}}_r) \frac{1}{|\widetilde{\mathbf{B}}|^2} d\varphi, \quad (3.14)$$

which mirrors the fact that the  $(\mathbf{e}_r, \mathbf{e}_\varphi)$ -coordinate system rotates itself with respect to the  $(\mathbf{e}_\zeta, \mathbf{e}_\rho)$ -system.

Rotation numbers already make sense for continuous fields; in fact, Lemma 3.1 holds for merely continuous fields as well, as can be shown by uniformly approximating  $\mathbf{B} \in C(\overline{G})$  by a sequence  $(B_n) \subset C^1(\overline{G})$  (cf. Heinz 1959, Definition 2).

An important property of the degree is its invariance under continuous deformations (cf. Heinz 1959, Theorem 3):

**Lemma 3.2 (deformation invariance)** Let  $G \subset \widehat{V}$  be a bounded domain,  $I$  be the closed interval  $[0, 1]$ , and  $\mathbf{B} : \overline{G} \times I \rightarrow \mathbb{R}^2$  be a continuous vector field which is  $\neq 0$  on  $\partial G \times I$ . Then

$$\deg(\mathbf{B}(\cdot, \lambda), G) = \text{const} \quad \text{for all } \lambda \in I.$$

Note that Lemma 3.2 implies, in particular, invariance of the rotation number  $\varrho(\mathbf{B}, c)$  under continuous deformations of  $\mathbf{B}$ .

REMARK: The rotation number of simple direction fields can be determined by just counting the “signed poles”. Let  $\mathbf{D} = D_r \mathbf{e}_r + D_\varphi \mathbf{e}_\varphi$ , parametrized with the polar angle  $\varphi$ , have  $l_{\mathbf{D}}$  isolated poles  $\{\varphi_n : n = 1, \dots, l_{\mathbf{D}}\}$ , i.e.  $D_\varphi(\varphi_n) = 0$  (cf. Section 1), then

$$\varrho(\mathbf{D}, c_1) = 1 + \frac{1}{2\pi} \sum_{n=1}^{l_{\mathbf{D}}} (\arctan D_r/D_\varphi) \Big|_{\varphi=\varphi_n^-}^{\varphi=\varphi_n^+}.$$

If the poles are “simple”, i.e.  $\frac{d}{d\varphi} D_\varphi|_{\varphi=\varphi_n} \neq 0$ , we can also write

$$\varrho(\mathbf{D}, c_1) = 1 + \frac{1}{2} \sum_{n=1}^{l_{\mathbf{D}}} \text{sign} \left[ \left( D_r \frac{d}{d\varphi} D_\varphi \right) \Big|_{\varphi=\varphi_n} \right]. \quad (3.15)$$

If  $\mathbf{B}$  resp.  $f$  are solutions of eqs. (3.1) resp. (3.7) we know more about the index of the zeroes. Useful is here the following representation theorem for  $f$  (cf. Bers et al. 1964, p.259)

**Lemma 3.3** *Let  $D \subset \mathbb{C}$  be the unit disc and  $G \subset D$  be a bounded domain. Let, furthermore,  $f : G \rightarrow \mathbb{C}$  be a (weak) solution of the differential equation*

$$\partial_{\bar{z}} f = \alpha f + \beta \bar{f} \quad (3.16)$$

*with bounded measurable coefficients  $\alpha, \beta : G \rightarrow \mathbb{C}$ . Then  $f$  admits the representation*

$$f(z, \bar{z}) = h(z) e^{g(z, \bar{z})}, \quad (3.17)$$

*where  $h$  is complex analytic in  $G$  and  $g$  uniformly Hölder continuous in  $D$ .*

REMARK: The restriction  $G \subset D$  is not essential.  $D$  may be replaced by another domain by means of a conformal transformation; in particular,  $D$  may be replaced by  $\widehat{V}$  using the inversion  $z \rightarrow 1/z$ .

The solutions of eq. (3.4) in  $\widehat{V}$  vanishing at spatial infinity are well-known (e.g. Folland 1995, p.144). They have a series representation

$$\psi(r, \varphi) = \psi^{(\tilde{\delta})}(r, \varphi) = \sum_{n=\tilde{\delta}-2}^{\infty} \frac{c_n}{r^{n+1}} P_n(\cos \varphi), \quad c_{\tilde{\delta}-2} \neq 0,$$

which converges uniformly and absolutely for any  $r > 1$ . Here  $P_n$  is the Legendre polynomial of order  $n$  and  $\tilde{\delta} \in \mathbb{N} \setminus \{1, 2\}$  is the exact decay order of the associated magnetic field. This reads

$$\begin{aligned} \tilde{\mathbf{B}}(r, \varphi) &= \tilde{\mathbf{B}}^{(\tilde{\delta})}(r, \varphi) = - \sum_{n=\tilde{\delta}-2}^{\infty} \frac{c_n}{r^{n+2}} (D_r^{(n)}(\varphi) \mathbf{e}_r + D_\varphi^{(n)}(\varphi) \mathbf{e}_\varphi) \\ &= - \frac{c_{\tilde{\delta}-2}}{r^{\tilde{\delta}}} \mathbf{D}^{(\tilde{\delta}-2)}(\varphi) + O(r^{-(\tilde{\delta}+1)}) \quad \text{for } r \rightarrow \infty \end{aligned} \quad (3.18)$$

with

$$\mathbf{D}^{(n)}(\varphi) := (n+1)P_n(\cos \varphi) \mathbf{e}_r + P_n'(\cos \varphi) \sin \varphi \mathbf{e}_\varphi \quad (3.19)$$

being the exterior axisymmetric  $2^n$ -pole field restricted on the unit circle.<sup>3</sup> Computing the rotation number of  $\mathbf{D}^{(n)}$  (e.g. with (3.15)) one finds  $\varrho(\mathbf{D}^{(n)}, c_1) = 1 + n$ . We see from (3.18)<sub>2</sub> that  $\tilde{\mathbf{B}}^{(\tilde{\delta})}$  has no zeroes beyond some radius  $R_0$ . So, with (3.12), (3.13) follows that  $\varrho(\mathbf{B}, c_R) = \text{const}$  for  $R > R_0$ . We obtain then with (3.18)

$$\varrho_\infty(\tilde{\mathbf{B}}^{(\tilde{\delta})}) := \lim_{R \rightarrow \infty} \varrho(\tilde{\mathbf{B}}^{(\tilde{\delta})}, c_R) = \varrho(\mathbf{D}^{(\tilde{\delta}-2)}, c_1) = \tilde{\delta} - 1. \quad (3.20)$$

Concerning the zeroes of  $\tilde{\mathbf{B}}$  we have the following

**Lemma 3.4** *An axisymmetric harmonic field  $\mathbf{B}$ , i.e.  $\mathbf{B}$  solves the system (3.1) in  $\widehat{V}$ , has only isolated zeroes  $\mathbf{x}_n$  of finite order  $-\iota(\mathbf{B}, \mathbf{x}_n)$ . In particular, the index of  $\mathbf{B}$  in any  $\mathbf{x}_n$  is always negative. Denoting with  $\nu(\mathbf{B}, \widehat{V})$  the number of zeroes (counted as often as its order indicates) we have*

$$\nu(\mathbf{B}, \widehat{V}) = - \sum_{n=1}^{\infty} \iota(\mathbf{B}, \mathbf{x}_n). \quad (3.21)$$

PROOF: We distinguish zeroes on the symmetry axis and those outside the symmetry axis. In the former case a zero  $(\zeta_0, 0)$  can be shifted in the origin without affecting eq. (3.2) governing the magnetic field. Solving eq. (3.2) in polar coordinates, i.e. eq. (3.4), in a disc  $D$  around zero yields for  $\psi$  the series representation

$$\psi(r, \varphi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos \varphi),$$

and hence for the associated magnetic field:

$$\tilde{\mathbf{B}}(r, \varphi) = \sum_{n=1}^{\infty} c_n r^{n-1} \check{\mathbf{D}}^{(n)}(\varphi) \quad (3.22)$$

with

$$\check{\mathbf{D}}^{(n)}(\varphi) := n P_n(\cos \varphi) \mathbf{e}_r - P'_n(\cos \varphi) \sin \varphi \mathbf{e}_\varphi$$

being the (interior) axisymmetric  $2^n$ -pole field restricted on the unit circle.  $\mathbf{B}(0, \cdot) = 0$  means there is  $l \in \mathbb{N}$  (the order of the zero) such that (3.22) takes the form

$$\tilde{\mathbf{B}}(r, \varphi) = \sum_{n=l+1}^{\infty} c_n r^{n-1} \check{\mathbf{D}}^{(n)}(\varphi) = c_{l+1} r^l \check{\mathbf{D}}^{(l+1)}(\varphi) + O(r^{l+1}) \quad \text{for } r \rightarrow 0, \quad c_{l+1} \neq 0. \quad (3.23)$$

From (3.23) one reads off that the zero is isolated, and with (3.15) one computes for the index:

$$\iota(\mathbf{B}, 0) = \text{deg}(\mathbf{B}, D) = \lim_{r \rightarrow 0} \varrho(\mathbf{B}, c_r) = \varrho(\check{\mathbf{D}}^{(l+1)}, c_1) = -l.$$

To describe zeroes outside the symmetry axis we take advantage of the representation (3.17) provided in Lemma 3.3. Let  $z_0$  be a zero of  $\mathbf{B}$  in  $\widehat{V} \setminus \{\rho = 0\}$  and  $D_r \Subset \widehat{V} \setminus \{\rho = 0\}$  a disc around  $z_0$  with radius  $r$ . Setting  $\alpha := -\beta := \frac{1}{2} \frac{1}{z - \bar{z}}$  the function  $f$  according to (3.6) is clearly

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<sup>3</sup>Prime means derivation with respect to the argument.

a solution of (3.16) in  $D_r$ . Thus,  $z_0$  is a zero of the analytic function  $h$ , hence isolated and of order (say)  $l$ . Computing the index of  $\mathbf{B}$  in  $z_0$  we find

$$\iota(\mathbf{B}, z_0) = \lim_{r \rightarrow 0} \varrho(\mathbf{B}, \partial D_r) = \lim_{r \rightarrow 0} \varrho(\overline{f}, \partial D_r) = - \lim_{r \rightarrow 0} \varrho(h, \partial D_r) = -l.$$

□

Summarizing the Lemmata 3.1 and 3.4 we have

**Lemma 3.5** *Let  $\mathbf{B} \in C^1(\widehat{V}) \cap C(\overline{\widehat{V}})$  be an axisymmetric harmonic field  $\neq 0$  on  $S^1$  with rotation number  $\varrho(\mathbf{B}, c_1) = \varrho$  and exact decay order  $\tilde{\delta}$ , then  $\mathbf{B}$  has a finite number  $\nu(\mathbf{B}, \widehat{V})$  of zeroes in  $\widehat{V}$  given by*

$$\nu(\mathbf{B}, \widehat{V}) = \varrho - \tilde{\delta} + 1. \quad (3.24)$$

PROOF: Noting that  $\mathbf{B}$  has no zeroes close to  $c_1$  and beyond some radius  $R_1$  eq. (3.24) follows from eqs. (3.12), (3.13), (3.20), and (3.21) in the limites  $R \rightarrow 1$  and  $\tilde{R} \rightarrow \infty$ .

□

Applying Lemma 3.5 on solutions of the direction problem yields a bound on the number of linear independent solutions:

**Theorem 3.1** *Let  $\mathbf{D}$  be a symmetric direction field with rotation number  $\varrho$  and let  $L_{3a}^\delta(\mathbf{D})$  be the space of all solutions of the unsigned problem  $P_{3a}^u$  with decay order  $\delta \in \mathbb{N} \setminus \{1, 2\}$ . Then*

$$\dim L_{3a}^\delta(\mathbf{D}) \leq \max\{\varrho - \delta + 2, 0\}. \quad (3.25)$$

PROOF: We consider the case  $\varrho > \delta - 2$  first. Let us assume that there are  $\varrho - \delta + 3$  linear independent solutions  $\mathbf{B}^{(1)}, \dots, \mathbf{B}^{(\varrho - \delta + 3)}$  of problem  $P_{3a}^u$  with representations

$$\tilde{\mathbf{B}}^{(n)}(r, \varphi) = - \sum_{m=\delta-2}^{\infty} \frac{c_m^n}{r^{m+2}} (D_r^{(m)}(\varphi) \mathbf{e}_r + D_\varphi^{(m)}(\varphi) \mathbf{e}_\varphi)$$

according to (3.18). Let  $\{\lambda_n : 0 \leq n \leq \varrho - \delta + 3\}$  be a nontrivial solution of the linear system

$$\sum_{n=1}^{\varrho - \delta + 3} \lambda_n c_m^n = 0, \quad m = \delta - 2, \dots, \varrho - 1.$$

Defining  $\mathbf{B} := \sum_{n=1}^{\varrho - \delta + 3} \lambda_n \mathbf{B}^{(n)}$ ,  $\mathbf{B}$  is still a solution of problem  $P_{3a}^u$  but with decay order  $\varrho + 2$ . If  $\mathbf{B}|_{c_1} \neq 0$  we can apply Lemma 3.5 and obtain the contradiction

$$\nu(\mathbf{B}, \widehat{V}) = \varrho - \tilde{\delta} + 1 \leq \varrho - (\varrho + 2) + 1 = -1.$$

Thus, we must drop our assumption.

If  $\mathbf{B}$  has zeroes on  $c_1$  we need some finer consideration: As  $\mathbf{B}$  is an axisymmetric harmonic field it has only isolated zeroes of finite order in  $\widehat{V}$ . These zeroes can not cluster at  $c_1$  since the direction field of  $\mathbf{B}$  rotates around a zero by an integer value  $\neq 0$ , whereas the direction field on  $c_1$  is *fixed*. So, we can choose some  $\varepsilon > 0$  such that  $\mathbf{B} \neq 0$  on  $A_\varepsilon \cup c_{1+\varepsilon}$ , where  $A_\varepsilon$  is the open annulus bounded by the circles  $c_1$  and  $c_{1+\varepsilon}$ . We show now

$$\varrho(\mathbf{B}, c_{1+\varepsilon}) = \varrho(\mathbf{D}, c_1) = \varrho; \quad (3.26)$$

so, applying Lemma 3.5 on  $\mathbf{B}$  in  $\widehat{V} \setminus A_\varepsilon$  yields again a contradiction.

In order to prove (3.26) we extend  $\mathbf{D}$  parallel to  $\mathbf{B}$  to a continuous field  $\mathbf{D}_{ex} \neq 0$  on  $\overline{A_\varepsilon}$ . Writing  $\mathbf{B} = a_{ex} \mathbf{D}_{ex}$  in  $\overline{A_\varepsilon}$  defines likewise a continuous extension of  $a$  onto  $\overline{A_\varepsilon}$ . Let now  $a(\cdot, \lambda) = a_\lambda$ ,  $\lambda \in [0, 1]$  be a continuous deformation between  $a_0 = a_{ex}$  and  $a_1 \equiv 1$ , and define

$$\mathbf{B}_\lambda := a_\lambda \mathbf{D}_{ex} + \mathbf{O}, \quad \lambda \in [0, 1],$$

where the continuous field  $\mathbf{O} \neq 0$  is chosen everywhere orthogonal to  $\mathbf{D}_{ex}$  in  $\overline{A_\varepsilon}$ . We have then clearly  $\mathbf{B}_\lambda \neq 0$  on  $\overline{A_\varepsilon} \times [0, 1]$  and the Lemmata 3.1 and 3.2 yield

$$\varrho(\mathbf{B} + \mathbf{O}, c_{1+\varepsilon}) = \varrho(a_{ex} \mathbf{D}_{ex} + \mathbf{O}, c_{1+\varepsilon}) = \varrho(\mathbf{B}_0, c_{1+\varepsilon}) = \varrho(\mathbf{B}_0, c_1) = \varrho(\mathbf{B}_1, c_1) = \varrho(\mathbf{D} + \mathbf{O}, c_1).$$

Replacing  $\mathbf{O}$  by  $\mu \mathbf{O}$  and using once more the deformation invariance of the rotation number, (3.26) is then obtained in the limit  $\mu \rightarrow 0$ .

Finally, considering the case  $\varrho \leq \delta - 2$ , we assume  $\mathbf{B}$  being a nontrivial solution  $\in L_{3a}^\delta(\mathbf{D})$ . If  $\mathbf{B}|_{c_1} \neq 0$  we can apply Lemma 3.5 and obtain the contradiction

$$\delta - 2 \geq \varrho \geq \tilde{\delta} - 1 \geq \delta - 1.$$

If  $\mathbf{B}$  has zeroes on  $c_1$  we make use again of (3.26) and a contradiction arises if Lemma 3.5 is applied on  $\mathbf{B}$  in  $\widehat{V} \setminus A_\varepsilon$ . So, in the case  $\varrho \leq \delta - 2$  we find  $L_{3a}^\delta(\mathbf{D}) = \{0\}$ .  $\square$

REMARK: The restriction of the boundary in the direction problem to a circle is not essential. Theorem 3.1 holds likewise if  $S^1$  is replaced by a simple closed symmetric  $C^1$ -curve.

### 3.2 Existence of solutions

In order to find signed solutions we use the complex formulation  $P_{3ac}^s$  of the signed problem and try to mimick the two-dimensional construction, i.e. we use the ansatz

$$f(z, \bar{z}) = h(z) e^{g(z, \bar{z})}, \quad z \in \widehat{V} \tag{3.27}$$

for the solution  $f$  with suitably chosen complex analytic function  $h$  and try to formulate a well-posed boundary-value problem for  $\Im g$ . We start again with computing the rotation number  $\varrho(f, c_r)$  along  $c_r \subset \widehat{V}$  under the assumptions  $h|_{c_r} \neq 0$  and  $g \in C^1$  in a neighbourhood of  $c_r$ :

$$\begin{aligned} \varrho(f, c_r) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{d}{d\varphi} [i \arg f(re^{i\varphi})] d\varphi \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{d}{d\varphi} [\log(h(re^{i\varphi}) \exp g(re^{i\varphi}, re^{-i\varphi}))] d\varphi \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{d}{d\varphi} [\log h(re^{i\varphi}) + g(re^{i\varphi}, re^{-i\varphi})] d\varphi = \frac{1}{2\pi i} \int_{c_r} \frac{h'(z)}{h(z)} dz. \end{aligned}$$

Thus, with Lemma 2.1 we have an argument principle for  $f$ :

$$\nu(f, A) = \nu(h, A) = \frac{1}{2\pi i} \int_{c_R} \frac{h'(z)}{h(z)} dz - \frac{1}{2\pi i} \int_{c_r} \frac{h'(z)}{h(z)} dz = \varrho(f, c_R) - \varrho(f, c_r), \tag{3.28}$$

where the annulus  $A$  is bounded by  $c_r$  and  $c_R$ ,  $1 < r < R$ . Applying (3.28) on a solution  $f$  of problem  $P_{3ac}^s$  with symmetric direction field  $\mathbf{D}$  with rotation number  $\varrho$  and exact decay order  $\tilde{\delta}$  suggests for  $h$  the ansatz

$$h(z) = \prod_{n=1}^{\varrho - \tilde{\delta} + 1} (z - z_n) z^{-\varrho} \quad (3.29)$$

with a symmetric<sup>4</sup> set of numbers  $\{z_1, \dots, z_{\varrho - \tilde{\delta} + 1}\} \subset \widehat{V}$ . In fact, evaluating  $\arg f$  on  $c_r$  we find with (3.10) in the limit  $r \rightarrow 1$ :

$$-\arg D_c = \arg f|_{c_1} = \sum_{n=1}^{\varrho - \tilde{\delta} + 1} \arg(z - z_n) - \varrho \arg z + \Im g|_{c_1}.$$

Thus, the boundary function (parametrized with the polar angle  $\varphi$ )

$$\phi(\varphi) := \Im g|_{c_1}(\varphi) = -\arg D_c(\varphi) - \sum_{n=1}^{\varrho - \tilde{\delta} + 1} \arg(e^{i\varphi} - z_n) + \varrho \varphi \quad (3.30)$$

satisfies  $\phi(2\pi) = \phi(0)$  and is hence well-defined and continuous on  $c_1$ . On the other side computing  $\varrho(f, c_R)$  for any  $R > \max\{|z_1|, \dots, |z_{\varrho - \tilde{\delta} + 1}|\}$  we obtain

$$\varrho(\bar{f}, c_R) = \varrho(\bar{h}, c_R) = -(\varrho - \tilde{\delta} + 1) + \varrho = \tilde{\delta} - 1,$$

which means in view of (3.6), (3.20) an exact decay order  $\tilde{\delta}$  of the corresponding field  $\mathbf{B}$ . Finally, inserting the ansatz (3.27) in eq. (3.7) yields

$$-\partial_{\bar{z}} g = \frac{1}{2} \frac{1}{z - \bar{z}} \left( \frac{\bar{h}}{h} e^{-2i\Im g} - 1 \right). \quad (3.31)$$

Differentiating (3.31) with respect to  $z$  we obtain a semilinear elliptic equation for  $q := 2\Im g$ :

$$-\partial_z \partial_{\bar{z}} q = \Im \left\{ \partial_z \left[ \frac{1}{z - \bar{z}} \left( \frac{\bar{h}}{h} e^{-iq} - 1 \right) \right] \right\}. \quad (3.32)$$

Once eq. (3.32) has been solved the real part  $p := 2\Re g$  can be determined up to a constant  $p_0$  from (3.31):

$$-\partial_{\bar{z}} p = i \partial_{\bar{z}} q + \frac{1}{z - \bar{z}} \left( \frac{\bar{h}}{h} e^{-iq} - 1 \right). \quad (3.33)$$

Using in  $\mathbb{R}^2$  the variables  $(\zeta, \rho)$  and the real notation  $\bar{h}/h =: v(\zeta, \rho) + iw(\zeta, \rho)$  eq. (3.32) takes the real form

$$-\Delta q + \frac{1}{\rho} \partial_{\zeta} (v \cos q + w \sin q) - \partial_{\rho} \left[ \frac{1}{\rho} (v \sin q - w \cos q) \right] = 0, \quad (3.34)$$

where  $\Delta := \partial_{\zeta}^2 + \partial_{\rho}^2$ . So, the problem is now to solve eq. (3.34) in  $\widehat{V}$  under the boundary condition  $q|_{c_1} = 2\phi$  with  $\phi$  given in (3.30) and the asymptotic condition  $q(\zeta, \rho) = O(1)$  for  $\zeta^2 + \rho^2 \rightarrow \infty$ .

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<sup>4</sup> $S$  is called symmetric if  $z \in S$  implies  $\bar{z} \in S$ .

If instead of  $\widehat{V}$  the bounded region  $A$  is considered the asymptotic condition is replaced by a second boundary condition, viz.  $q|_{c_R} = 2\widehat{\phi}$  with

$$\widehat{\phi}(\varphi) := -\arg \widehat{D}_c(\varphi) - \sum_{n=1}^{\varrho-\tilde{\delta}+1} \arg(e^{i\varphi} - z_n) + \varrho\varphi, \quad (3.35)$$

where the “exterior” direction field  $\widehat{\mathbf{D}} \in C(c_R, \mathbb{R}^2)$  has the rotation number  $\widehat{\varrho} := \tilde{\delta} - 1$ .

The main problem with eq. (3.34) is clearly the unboundedness of the coefficients on the symmetry axis  $\{\rho = 0\}$ . Embedding eq. (3.34) in  $\mathbb{R}^5$ , however, turns out to eliminate the “coordinate singularity” at  $\rho = 0$ . Introducing  $\tilde{q} := q/\rho$  we rewrite for this purpose eq. (3.34) in the form

$$\begin{aligned} -\left(\partial_\rho^2 + \frac{3}{\rho}\partial_\rho + \partial_\zeta^2\right)\tilde{q} + \frac{1}{\rho^2}\partial_\zeta(v \cos \rho\tilde{q} + w \sin \rho\tilde{q}) - \frac{1}{\rho^3}\partial_\rho[\rho(v \sin \rho\tilde{q} - w \cos \rho\tilde{q} - \rho\tilde{q})] \\ + \frac{2}{\rho^3}(v \sin \rho\tilde{q} - w \cos \rho\tilde{q} - \rho\tilde{q}) = 0. \end{aligned} \quad (3.36)$$

$\tilde{q}$  is now considered as an axisymmetric function in  $\mathbb{R}^5$  with symmetry axis in  $x_5$ -direction. Identifying  $\rho^2$  with  $\sum_{i=1}^4 x_i^2$  and  $\zeta$  with  $x_5$ ,  $\mathbf{x} \in \mathbb{R}^5$ , and defining  $u(\mathbf{x}) := \tilde{q}(\zeta, \rho)$  as well as

$$\begin{aligned} A_i(u, \mathbf{x}) &:= \frac{1}{\rho^2}(v \sin \rho u - w \cos \rho u - \rho u)\frac{x_i}{\rho}, \quad i = 1, \dots, 4, \\ A_5(u, \mathbf{x}) &:= -\frac{1}{\rho^2}(v \cos \rho u + w \sin \rho u), \\ B(u, \mathbf{x}) &:= \frac{2}{\rho^3}(v \sin \rho u - w \cos \rho u - \rho u) \end{aligned}$$

eq. (3.36) takes in  $\mathbb{R}^5$  the form

$$-\Delta u - \nabla \cdot \mathbf{A} + B = 0. \quad (3.37)$$

To see that  $\mathbf{A}$ ,  $B$  are indeed well-defined on the symmetry axis let us note some properties of  $v$  and  $w$ . We have by definition

$$v(\zeta, \rho) = \frac{1}{2} \frac{\overline{h^2}(z) + h^2(z)}{|h(z)|^2}, \quad w(\zeta, \rho) = \frac{1}{2i} \frac{\overline{h^2}(z) - h^2(z)}{|h(z)|^2}$$

with  $h$  of the form (3.29) with symmetrically distributed zeroes in  $\widehat{V}$ . We assume in the following no zero lying on the symmetry axis  $\{\rho = 0\}$ . From the definition follow the properties

$$v^2 + w^2 = 1 \quad \text{in } \widehat{V}, \quad v(\cdot, 0) = 1, \quad w(\cdot, 0) = 0,$$

as well as the symmetry relations

$$v(\zeta, \rho) = v(\zeta, -\rho), \quad w(\zeta, \rho) = -w(\zeta, -\rho).$$

Note that  $v$  and  $w$  are discontinuous in the zeroes of  $h$ . So,  $v$  and  $w$  are globally bounded but have no further global regularity. In a neighbourhood  $\{|\rho| < \varepsilon\}$  of the symmetry axis, however,  $v$  and  $w$  are real analytic and allow the expansions

$$\left. \begin{aligned} v(\zeta, \rho) &= 1 + v_2(\zeta)\rho^2 + O(\rho^4) \\ w(\zeta, \rho) &= w_1(\zeta)\rho + w_3(\zeta)\rho^3 + O(\rho^5) \end{aligned} \right\} |\rho| < \varepsilon, \quad (\zeta, \rho) \in \widehat{V}$$

with bounded coefficients  $v_2$ ,  $w_1$ , and  $w_3$ .

Rewriting the coefficients  $\mathbf{A}$  and  $B$  in the form

$$\begin{aligned}
A_i(u, \mathbf{x}) + w_1(x_5) \frac{x_i}{\rho^2} &= -\frac{w - w_1\rho}{\rho^2} \frac{x_i}{\rho} + \frac{v - 1}{\rho} \frac{x_i}{\rho} u + v \frac{\sin \rho u - \rho u}{\rho^2 u^2} \frac{x_i}{\rho} u^2 \\
&=: -a_i^{(0)}(\mathbf{x}) + a_i^{(1)}(\mathbf{x}) u - A_i^{(2)}(u, \mathbf{x}) u^2, \quad i = 1, \dots, 4, \\
A_5(u, \mathbf{x}) + \frac{1}{\rho^2} &= -\frac{v - 1}{\rho^2} - \frac{w}{\rho} u - \left( v \frac{\cos \rho u - 1}{\rho^2 u^2} + w \frac{\sin \rho u - \rho u}{\rho^2 u^2} \right) u^2 \\
&=: -a_5^{(0)}(\mathbf{x}) + a_5^{(1)}(\mathbf{x}) u - A_5^{(2)}(u, \mathbf{x}) u^2, \\
B(u, \mathbf{x}) + w_1(x_5) \frac{2}{\rho^2} &= -2 \frac{w - w_1\rho}{\rho^3} + 2 \frac{v - 1}{\rho^2} u \\
&\quad - 2 \frac{w}{\rho} \frac{\cos \rho u - 1}{\rho^2 u^2} u^2 + 2 v \frac{\sin \rho u - \rho u}{\rho^3 u^3} u^3 \\
&=: -b^{(0)}(\mathbf{x}) + b^{(1)}(\mathbf{x}) u - B^{(2)}(u, \mathbf{x}) u^2 - B^{(3)}(u, \mathbf{x}) u^3,
\end{aligned}$$

eq. (3.37) takes the form

$$-\Delta u - \nabla \cdot (\mathbf{a}^{(1)} u) + b^{(1)} u = -\nabla \cdot \mathbf{a}^{(0)} + b^{(0)} - \nabla \cdot (\mathbf{A}^{(2)} u^2) + B^{(2)} u^2 + B^{(3)} u^3 \quad (3.38)$$

with bounded coefficients  $\mathbf{a}^{(0)}$ ,  $\mathbf{a}^{(1)}$ ,  $\mathbf{A}^{(2)}$  and  $b^{(0)}$ ,  $b^{(1)}$ ,  $B^{(2)}$ ,  $B^{(3)}$ . Equation (3.38) is now well-defined on the symmetry axis but at the expense of nonlinear terms which are unbounded in  $u$ . The equation is not in variational form nor seem monotonicity methods be applicable since the coefficients in (3.38) do not observe any sign condition. The subsequent existence theorem about weak solutions in bounded domains for small data is based on fixed point techniques in  $L^p$ -spaces.

Let in the following  $G \subset \mathbb{R}^d$  be a bounded domain with  $C^1$ -boundary  $\partial G$  and  $W^{m,p}(G)$ ,  $W_0^{m,p}(G)$  with  $1 < p < \infty$  be the usual Sobolev spaces with norm  $\|\cdot\|_{m,p}$ ; in particular, we have  $W^{0,p}(G) = L^p(G)$  with norm  $\|\cdot\|_p$  and, in the case  $p = 2$ ,  $L^2(G)$  equipped with the scalar product  $(\cdot, \cdot)$ .  $S^p$  denotes the unit sphere in  $W_0^{1,p}(G)$ , i.e.  $\{v \in W_0^{1,p}(G) : \|v\|_{1,p} = 1\}$ .  $W^{-1,p}(G)$  is the dual space of  $W_0^{1,p'}(G)$ ,  $1/p + 1/p' = 1$ , with the dual pairing denoted by  $\langle f, v \rangle$ ,  $f \in W^{-1,p}(G)$ ,  $v \in W_0^{1,p'}(G)$ , and with norm  $\|f\|_{-1,p} = \sup_{v \in S^{p'}} \langle f, v \rangle$ .

We consider the boundary-value problem

$$\left. \begin{aligned}
L u &= f + N(u) && \text{in } G, \\
u &= \chi && \text{on } \partial G,
\end{aligned} \right\} \quad (3.39)$$

where  $L u := -\Delta u - \nabla \cdot (\mathbf{a}^{(1)} u) + b^{(1)} u$ ,  $f := -\nabla \cdot \mathbf{a}^{(0)} + b^{(0)}$ , and

$$N(u) := -\nabla \cdot \left( \sum_{i=2}^k \mathbf{A}^{(i)}(u, \cdot) u^i \right) + \sum_{j=2}^l B^{(j)}(u, \cdot) u^j \quad (3.40)$$

with  $k, l \in \mathbb{N} \setminus \{1\}$  and with bounded coefficients:

$$\left. \begin{aligned}
|\mathbf{a}^{(0)}|, |\mathbf{a}^{(1)}|, |b^{(0)}|, |b^{(1)}| &\leq M && \text{in } G, \\
|\mathbf{A}^{(i)}|, |\partial_u \mathbf{A}^{(i)}|, |B^{(j)}|, |\partial_u B^{(j)}| &\leq M && \text{in } \mathbb{R} \times G, \quad i = 2, \dots, k, j = 2, \dots, l.
\end{aligned} \right\} \quad (3.41)$$

If  $\chi$  is the trace of some function  $v \in W^{1,p}(G)$  we call a function  $u \in W^{1,p}(G)$  weak solution of problem (3.39) if  $N(u) \in W^{-1,p}(G)$  and if  $\tilde{u} := u - v \in W_0^{1,p}(G)$  satisfies

$$(\nabla \tilde{u}, \nabla \varphi) + (\mathbf{a}^{(1)} \tilde{u}, \nabla \varphi) + (b^{(1)} \tilde{u}, \varphi) = \langle f - Lv, \varphi \rangle + \langle N(\tilde{u} + v), \varphi \rangle \quad (3.42)$$

for any  $\varphi \in W_0^{1,p'}(G)$ .

Useful for the solution of the linearized problem is the following  $W^{1,p}$ -version of the Lax-Milgram Theorem, which is due to Simader (1972, p.97f):

**Lemma 3.6** *Let  $B : W_0^{1,p}(G) \times W_0^{1,p'}(G) \rightarrow \mathbb{R}$ , where  $1/p + 1/p' = 1$  and  $G \Subset \mathbb{R}^d$ ,  $d \geq 2$ ,  $\partial G \in C^1$ , be a continuous bilinear form. Let, furthermore,  $f \in W^{-1,p}(G)$ . If there are constants  $C, \hat{C} > 0$  such that*

$$C \|u\|_{1,p} \leq \sup_{\varphi \in S^{p'}} |B[u, \varphi]| \quad \text{for any } u \in W_0^{1,p}(G) \quad (3.43)$$

and

$$\hat{C} \|\varphi\|_{1,p'} \leq \sup_{u \in S^p} |B[u, \varphi]| \quad \text{for any } \varphi \in W_0^{1,p'}(G), \quad (3.44)$$

then there exists a unique function  $u \in W_0^{1,p}$  satisfying

$$B[u, \varphi] = \langle f, \varphi \rangle \quad \text{for any } \varphi \in W_0^{1,p'}(G). \quad (3.45)$$

Defining

$$B[u, \varphi] := (\nabla u, \nabla \varphi) + (a^{(1)} u, \nabla \varphi) + (b^{(1)} u, \varphi) \quad (3.46)$$

the weak formulation of problem (3.39) with  $N = 0$  reads

$$B[\tilde{u}, \varphi] = \langle g, \varphi \rangle \quad \text{for any } \varphi \in W_0^{1,p'}(G) \quad (3.47)$$

with  $\tilde{u} := u - v$  and  $g := f - Lv$ . Applying Lemma 3.6 on (3.47) yields

**Lemma 3.7** *Let  $\chi$  be the trace of some function  $v \in W^{1,p}(G)$ ,  $f \in W^{-1,p}(G)$ , and  $g := f - Lv \in W^{-1,p}(G)$ . The linearized problem (3.39) has then a unique weak solution  $u_{lin} \in W^{1,p}(G)$  satisfying the estimate*

$$\|u_{lin}\|_{1,p} \leq \|v\|_{1,p} + 2C_S \|g\|_{-1,p} \quad (3.48)$$

with some constant  $C_S = C_S(d, p, G) > 0$ , provided the coefficients  $a^{(1)}, b^{(1)}$  are sufficiently small, i.e.

$$|\mathbf{a}^{(1)}|, |b^{(1)}| \leq M \leq (4C_S)^{-1}. \quad (3.49)$$

PROOF: Equation (3.46) defines clearly a continuous bilinear form on  $W_0^{1,p}(G) \times W_0^{1,p'}(G)$ . So, to apply Lemma 3.6, it suffices to check the estimates (3.43) and (3.44). In the ‘‘unperturbed’’ case  $L = \Delta$  estimates of this type are well known (cf. Simader and Sohr 1996, p.44):

$$\|\nabla u\|_p \leq \tilde{C}_S \sup_{0 \neq \varphi \in W_0^{1,p'}(G)} \frac{(\nabla u, \nabla \varphi)}{\|\nabla \varphi\|_{p'}}. \quad (3.50)$$

Combining (3.50) with Poincaré’s inequality in the form

$$\|u\|_{1,p} \leq C_P \|\nabla u\|_p, \quad u \in W_0^{1,p}(G) \quad (3.51)$$

yields

$$\|u\|_{1,p} \leq C_S \sup_{\varphi \in S^{p'}} (\nabla u, \nabla \varphi) \quad (3.52)$$

with constants  $C_S, \tilde{C}_S, C_P > 0$  depending on  $d, p$ , and  $G$ .

With Hölder's inequality  $B[\cdot, \cdot]$  can be estimated as follows

$$|B[u, \varphi]| \geq |(\nabla u, \nabla \varphi)| - 2M \|u\|_p \|\varphi\|_{1,p'}.$$

Thus, with (3.52) and (3.49) we obtain

$$\sup_{\varphi \in S^{p'}} |B[u, \varphi]| \geq \frac{1}{C_S} \|u\|_{1,p} - \frac{1}{2C_S} \|u\|_{1,p} = \frac{1}{2C_S} \|u\|_{1,p}, \quad (3.53)$$

which is (3.43). An analogous estimate yields (3.44). Applying Lemma 3.6 on eq. (3.47) yields a unique function  $\tilde{u} \in W_0^{1,p'}(G)$  and  $u_{lin} := \tilde{u} + v$  is then the unique weak solution of problem (3.39) with  $N = 0$ . The estimate (3.48) follows with (3.53).  $\square$

For the nonlinear problem we have the following small-data result:

**Theorem 3.2** *Let  $p > d \geq 2$  and  $\chi$  being the trace of some function  $v \in W^{1,p}(G)$  with  $\|v\|_{1,p} < M$ . Then problem (3.39) with bounded coefficients according to (3.41) has a unique weak solution  $u \in W^{1,p}(G)$ , provided  $M$  is sufficiently small (depending on  $d, p, k, l$ , and  $G$ ).*

PROOF: Defining  $\hat{u} := u - u_{lin} = u - \tilde{u} - v$  eq. (3.42) takes the form

$$B[\hat{u}, \varphi] = \langle N(\hat{u} + u_{lin}), \varphi \rangle \quad (3.54)$$

with  $N$  given in (3.40). We prove now  $N(u) \in W^{-1,p}(G)$  for  $u \in W^{1,p}(G)$ ,  $p > d$ , and start with estimating  $(\mathbf{A}^{(i)} u^i, \nabla \varphi)$ : Setting  $r := ip$  we obtain with (3.41) and Hölder's inequality

$$|(\mathbf{A}^{(i)}(u, \cdot) u^i, \nabla \varphi)| \leq M \|u\|_r^i \|\nabla \varphi\|_{p'}. \quad (3.55)$$

Furthermore, choosing  $\tilde{p} \in [\frac{i-1}{i}d, d)$  Sobolev's embedding theorems yield constants  $C_1, C_2 > 0$  such that

$$\|u\|_r \leq C_1 \|u\|_{1,\tilde{p}} \leq C_2 \|u\|_{1,p}. \quad (3.56)$$

So, combining (3.55) and (3.56), and setting  $\tilde{p} := \frac{k-1}{k}d$  we obtain

$$\sup_{\varphi \in S^{p'}} \left| \left( \sum_{i=2}^k \mathbf{A}^{(i)}(u, \cdot) u^i, \nabla \varphi \right) \right| \leq C_3 M \sum_{i=2}^k \|u\|_{1,p}^i \quad (3.57)$$

with some constant  $C_3 = C_3(d, p, k, G) > 0$ .

Similarly, with  $\frac{j}{q} + \frac{1}{q'} = 1$ ,  $\frac{j}{q} \leq \frac{1}{p} + \frac{1}{d}$ , and  $\hat{p} \in [\frac{j-1}{j+1}d, d)$  the term  $(B^{(j)} u^j, \varphi)$  is estimated as follows:

$$|(B^{(j)}(u, \cdot) u^j, \varphi)| \leq M \|u\|_q^j \|\varphi\|_{q'} \leq C_4 M \|u\|_{1,\hat{p}}^j \|\varphi\|_{1,p'} \leq C_5 M \|u\|_{1,p}^j \|\varphi\|_{1,p'}.$$

Thus, setting  $\hat{p} := \frac{l-1}{l+1}d$  we obtain for the second term in  $N(u)$ :

$$\sup_{\varphi \in S^{p'}} \left| \left( \sum_{j=2}^l B^{(j)}(u, \cdot) u^j, \varphi \right) \right| \leq C_6 M \sum_{j=2}^l \|u\|_{1,p}^j, \quad (3.58)$$

which proves the assertion.

Let now  $u_{lin} \in W^{1,p}(G)$  be the linearized solution according to Lemma 3.7 and  $w \in W_0^{1,p}(G)$  arbitrary. The unique solution of eq. (3.45) with  $f := N(w + u_{lin})$  according to Lemma 3.6 defines then a continuous linear mapping  $T : W_0^{1,p}(G) \rightarrow W_0^{1,p}(G)$  with bound

$$\|Tw\|_{1,p} \leq 2C_S M \left( C_3 \sum_{i=2}^k \|w + u_{lin}\|_{1,p}^i + C_6 \sum_{j=2}^l \|w + u_{lin}\|_{1,p}^j \right). \quad (3.59)$$

In fact, choosing  $w$  and  $u_{lin}$  in the  $K$ -ball

$$B_K^p := \{w \in W_0^{1,p}(G) : \|w\|_{1,p} < K\},$$

where  $K \leq 1$ , inequality (3.59) can be further estimated by

$$\|Tw\|_{1,p} \leq C_7 M (\|w\|_{1,p}^2 + \|u_{lin}\|_{1,p}^2)$$

with some constant  $C_7 = C_7(d, p, k, l, G) > 0$ . So, with

$$K \leq \min\{1, (2C_7M)^{-1}\} \quad (3.60)$$

$T$  maps the closed ball  $\overline{B_K^p}$  onto itself.

Next we prove  $T$  to be contracting on  $\overline{B_K^p}$ , which ensures the existence of a fixed point of  $T$  solving, finally, eq. (3.54). So, we have to show

$$\|Tw - T\tilde{w}\|_{1,p} \leq L \|w - \tilde{w}\|_{1,p}$$

for some  $L < 1$  and any  $w, \tilde{w} \in B_K^p$ , or using (3.53)

$$\sup_{\varphi \in S^{p'}} |\langle N(w + u_{lin}) - N(\tilde{w} + u_{lin}), \varphi \rangle| \leq \frac{L}{2C_S} \|w - \tilde{w}\|_{1,p}. \quad (3.61)$$

With the abbreviation  $\mathbf{A}^{(i)}(w + u_{lin}, \mathbf{x})(w + u_{lin})^i =: \mathcal{A}^{(i)}(w, \mathbf{x})$  we can write

$$\mathcal{A}^{(i)}(w, \mathbf{x}) - \mathcal{A}^{(i)}(\tilde{w}, \mathbf{x}) = \partial_w \mathcal{A}^{(i)}(w_\theta, \mathbf{x})(w - \tilde{w}),$$

where  $w_\theta := (1 - \theta)w + \theta\tilde{w}$  with some  $\theta \in [0, 1]$ . Setting  $r := ip$ ,  $\tilde{r} := (i + 1)p$  we obtain with (3.41) and Hölder's inequality

$$|(\mathcal{A}^{(i)}(w, \cdot) - \mathcal{A}^{(i)}(\tilde{w}, \cdot), \nabla \varphi)| \leq M (\|w_\theta + u_{lin}\|_{\tilde{r}}^i \|w - \tilde{w}\|_{\tilde{r}} + i \|w_\theta + u_{lin}\|_{\tilde{r}}^{i-1} \|w - \tilde{w}\|_r) \|\nabla \varphi\|_{p'}.$$

Proceeding now as in the estimate (3.56) and setting  $\tilde{p} := \frac{k}{k+1}d$  we arrive at

$$\begin{aligned} \sup_{\varphi \in S^{p'}} \left| \left( \sum_{i=2}^k (\mathcal{A}^{(i)}(w, \cdot) - \mathcal{A}^{(i)}(\tilde{w}, \cdot)), \nabla \varphi \right) \right| \\ \leq C_8 M \sum_{i=2}^k (\|w_\theta + u_{lin}\|_{1,p}^i + \|w_\theta + u_{lin}\|_{1,p}^{i-1}) \|w - \tilde{w}\|_{1,p} \\ \leq C_9 M (\|w\|_{1,p} + \|\tilde{w}\|_{1,p} + \|u_{lin}\|_{1,p}) \|w - \tilde{w}\|_{1,p}, \end{aligned}$$

where we used  $\|w_\theta\|_{1,p} \leq \|w\|_{1,p} + \|\tilde{w}\|_{1,p} \leq 2K \leq 2$  and  $\|u_{lin}\|_{1,p} \leq K \leq 1$ ; the positive constant  $C_9$  depends on  $d, p, k$ , and  $G$ . This is the first part in (3.61).

With  $B^{(j)}(w + u_{lin}, \mathbf{x})(w + u_{lin})^j =: \mathcal{B}^{(j)}(w, \mathbf{x})$  we obtain analogously for the second part

$$\sup_{\varphi \in S^{p'}} \left| \left( \sum_{j=2}^l (\mathcal{B}^{(j)}(w, \cdot) - \mathcal{B}^{(j)}(\tilde{w}, \cdot)), \varphi \right) \right| \leq C_{10} M (\|w\|_{1,p} + \|\tilde{w}\|_{1,p} + \|u_{lin}\|_{1,p}) \|w - \tilde{w}\|_{1,p}$$

with some constant  $C_{10} = C_{10}(d, p, l, G) > 0$ .

So, with the further condition

$$K \leq (6 C_S (C_9 + C_{10}) M)^{-1} \quad (3.62)$$

the mapping  $T$  is indeed contracting on  $\overline{B_K^p}$ .

Note, finally, that with (3.48), (3.41), and  $\|v\|_{1,p} \leq M$  the condition  $u_{lin} \in B_K^p$  amounts to

$$C_{11} M + C_{12} M^2 < K \quad (3.63)$$

with some constants  $C_{11}, C_{12}$  depending on  $d, p$ , and  $G$ . Therefore, the smallness assumption on  $M$  is encoded in, besides condition (3.49), the conditions (3.60), (3.62), and (3.63).  $\square$

REMARK: According to Sobolev's embedding theorems we have  $W^{1,p}(G) \subset C^{0,1-d/p}(\overline{G})$ , i.e. our weak solution is Hölder continuous with exponent  $1 - d/p$  in  $G$  up to the boundary.

Interestingly, the regularity of the boundary function  $\chi$  can likewise be characterized in terms of Hölder continuity though with a larger exponent:  $\chi$  being the trace of a  $W^{1,p}$ -function means  $\chi \in W^{1-1/p,p}(\partial G)$  (cf. Adams 1975, p.217). Elements of the "trace space"  $W^{1-1/p,p}(\partial G)$  are characterized by the finite norm condition (cf. Galdi 1994, p.43)

$$\|\chi\|_{L^p(\partial G)}^p + \int_{\partial G} \int_{\partial G} \frac{|\chi(\mathbf{x}) - \chi(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{d+p-2}} ds(\mathbf{x}) ds(\mathbf{y}) < \infty, \quad (3.64)$$

where  $ds$  denotes the  $(d-1)$ -dimensional volume element on  $\partial G$ . Condition (3.64) is clearly satisfied if  $\chi$  is Hölder continuous of class  $C^{0,\gamma}(\partial G)$  with  $\gamma > 1 - 1/p$ .

Theorem 3.2 can now be used to construct solutions of problem  $P_{3ac}^s(A)$ . Let for this purpose  $d = 5, k = 2, l = 3$ , and  $G = A_5 \subset \mathbb{R}^5$  be the spherical shell bounded by spheres  $S$  and  $\widehat{S}$  with radii 1 and  $R$ , respectively. Let, furthermore,  $u \in W^{1,6}(A)$  be an axisymmetric solution of problem (3.39) with axisymmetric boundary functions  $\chi$  and  $\widehat{\chi}$  on  $S$  and  $\widehat{S}$ , respectively. Going back to the two-dimensional setting and using the variables  $(\zeta, \rho)$  in  $\mathbb{R}^2$  the function  $q(\zeta, \rho)/\rho := u(\mathbf{x}) \in W^{1,6}(A) \cap C^{0,1/6}(\overline{A})$  is then a symmetric solution of eq. (3.34) in the annulus  $A \subset \mathbb{R}^2$  corresponding to  $A_5 \subset \mathbb{R}^5$  with symmetric boundary functions  $\phi = \rho\chi/2$  and  $\widehat{\phi} = \rho\widehat{\chi}/2$ . Writing eq. (3.33) with variables  $(\zeta, \rho)$  we find

$$\left. \begin{aligned} \partial_\zeta p &= \partial_\rho q + v \frac{\sin q}{q} \frac{q}{\rho} - \frac{w}{\rho} \cos q, \\ \partial_\rho p &= -\partial_\zeta q + v \frac{\cos q - 1}{q} \frac{q}{\rho} + \frac{w}{\rho} \sin q. \end{aligned} \right\} \quad (3.65)$$

Note that the right-hand side of eq. (3.65) is in  $L^6(A)$ . Equation (3.65) has then a solution  $p \in W^{1,6}(A) \cap C^{0,1/6}(\overline{A})$ , and setting (in complex notation)  $f(z, \bar{z}) := h(z) \exp(p(\zeta, \rho) + iq(\zeta, \rho))$

we find  $f \in W^{1,6}(A)$  solving eq. (3.7) a.e. in  $A$ . With the correspondence (3.6)  $f$  is equivalent to a field  $\mathbf{B}$  satisfying eqs. (3.1) a.e. in  $A$ , which means  $\mathbf{B}$  to be an axisymmetric (weakly) harmonic field in  $A_3 := A \times \{0 \leq \theta < 2\pi\} \subset \mathbb{R}^3$ . Higher interior regularity of  $\mathbf{B}$  and hence  $f$  is then obtained by standard arguments (cf. Gilbarg and Trudinger 1998, p.183f). Finally, since  $f$  is continuous on  $\bar{A}$  we find with (3.30) and (3.35)  $\arg f|_{c_1} = -\arg D_c$  and  $\arg f|_{c_R} = -\arg \widehat{D}_c$ , respectively. So, the solution of problem (3.39) provides indeed a solution of the bounded version  $P_{3ac}^s(A)$  of the signed direction problem. Unfortunately, given the direction fields and a (symmetric) distribution of zeroes in  $A$  it seems to be difficult to verify the smallness assumptions required by Theorem 3.2. On the premises that these assumptions can be satisfied we have

**Corollary 3.1** *Let  $A \subset \mathbb{R}^2$  be the annulus bounded by the circles  $c_1$  and  $c_R$ ,  $R > 1$ , let  $\mathbf{D}$  and  $\widehat{\mathbf{D}}$  be direction fields along  $c_1$  and  $c_R$  with rotation numbers  $\varrho$  and  $\hat{\varrho} = \tilde{\delta} - 1$ ,  $\hat{\varrho} \leq \varrho$ , respectively, and let  $\{z_1, \dots, z_{\varrho - \hat{\varrho}}\} \subset A$  be a symmetric set of numbers. Let, furthermore,  $v \in W^{1,6}(A)$  with traces  $v|_{c_1} = 2\phi/\rho$ ,  $v|_{c_R} = 2\widehat{\phi}/\rho$  and  $\phi, \widehat{\phi}$  given by eqs. (3.30), (3.35), respectively. Finally, let the smallness conditions in Theorem 3.2 with  $d = 5$ ,  $p = 6$ ,  $k = 2$ , and  $l = 3$  be satisfied. Then problem  $P_{3ac}^s(A)$  has a unique solution  $f \in C^1(A) \cap C(\bar{A})$  with the  $\varrho - \hat{\varrho}$  zeroes  $z_1, \dots, z_{\varrho - \hat{\varrho}}$ .*

Note that the solution of the corollary is uniquely determined. All the nonuniqueness of the direction problem is encoded in the arbitrary choice of the zeroes. This freedom is in fact sufficient to provide enough solutions of the unsigned direction problem so that the upper bound on the solution space (3.25) is sharp. To see this we switch (for reasons of comparison with (3.25)) back to the unbounded region  $\widehat{V}$ . Let  $\mathbf{B}^{(\tilde{\delta})}$  be a signed solution of the direction problem with exact decay order  $\tilde{\delta}$ . According to Lemma 3.5,  $\mathbf{B}^{(\tilde{\delta})}$  has exactly  $\varrho - \tilde{\delta} + 1$  zeroes in  $\widehat{V}$ . It is then not hard to see that any set of signed solutions  $\{\mathbf{B}^{(\tilde{\delta})} : \tilde{\delta} = \delta, \dots, \varrho + 1\}$  is a basis of the solution space. If zeroes on the symmetry axis are not allowed (as in Theorem 3.2) the situation is slightly trickier since only solutions with an even number of (symmetrically distributed) zeroes are at our disposal. In that case we need two ‘‘asymptotically independent’’ solutions for a given (even) number of zeroes. Two axisymmetric harmonic fields  $\mathbf{B}^{(\tilde{\delta})}$  and  $\widetilde{\mathbf{B}}^{(\tilde{\delta})}$  with exact decay orders  $\tilde{\delta}$  and representations according to (3.18),

$$\mathbf{B}^{(\tilde{\delta})}(r, \varphi) = \sum_{n=\tilde{\delta}-2}^{\infty} \frac{c_n^{\tilde{\delta}}}{r^{n+2}} \mathbf{D}^{(n)}(\varphi), \quad \widetilde{\mathbf{B}}^{(\tilde{\delta})}(r, \varphi) = \sum_{n=\tilde{\delta}-2}^{\infty} \frac{\tilde{c}_n^{\tilde{\delta}}}{r^{n+2}} \mathbf{D}^{(n)}(\varphi)$$

are called asymptotically independent, if their two lowest-order coefficients satisfy the condition

$$c_{\tilde{\delta}-2}^{\tilde{\delta}} \tilde{c}_{\tilde{\delta}-1}^{\tilde{\delta}} - c_{\tilde{\delta}-1}^{\tilde{\delta}} \tilde{c}_{\tilde{\delta}-2}^{\tilde{\delta}} \neq 0.$$

In this situation we have

**Theorem 3.3** *Let  $\mathbf{D}$  be a symmetric direction field with rotation number  $\varrho$  and let  $L_{3a}^{\delta}(\mathbf{D})$  be the space of all solutions of the unsigned problem  $P_{3a}^u$  with decay order  $\delta \in \mathbb{N} \setminus \{1, 2\}$ . Let  $\varrho > \delta - 2$ . If there exist a signed solution  $\mathbf{B}^{(\varrho+1)}$  of the direction problem without zero and asymptotically independent solutions  $\mathbf{B}^{(\varrho+1-2\mu)}$ ,  $\widetilde{\mathbf{B}}^{(\varrho+1-2\mu)}$  with exactly  $2\mu$  zeroes,  $\mu = 1, \dots, [(\varrho - \delta)/2] + 1$ <sup>5</sup>, then*

$$\dim L_{3a}^{\delta}(\mathbf{D}) = \varrho - \delta + 2.$$

<sup>5</sup> ‘‘ $[x]$ ’’ denotes here the largest integer value  $\leq x$ .

PROOF: Let us first assume  $\varrho + 1 - \delta$  to be even. In this case the  $\varrho - \delta + 2$  functions  $\mathbf{B}^{(\varrho+1)}$ ,  $\mathbf{B}^{(\varrho+1-2\mu)}$ , and  $\tilde{\mathbf{B}}^{(\varrho+1-2\mu)}$ ,  $\mu = 1, \dots, (\varrho + 1 - \delta)/2$  have all decay order  $\delta$  and are linear independent, which can be seen as in the proof of Lemma 2.5: Inserting the representations

$$\left. \begin{aligned} \mathbf{B}^{(\varrho+1)}(r, \varphi) &= \sum_{n=\varrho-1}^{\infty} \frac{c_n^{\varrho+1}}{r^{n+2}} \mathbf{D}^{(n)}(\varphi), & c_{\varrho-1}^{\varrho+1} &\neq 0, \\ \mathbf{B}^{(\varrho+1-2\mu)}(r, \varphi) &= \sum_{n=\varrho-1-2\mu}^{\infty} \frac{c_n^{\varrho+1-2\mu}}{r^{n+2}} \mathbf{D}^{(n)}(\varphi), & c_{\varrho-1-2\mu}^{\varrho+1-2\mu} &\neq 0, \\ \tilde{\mathbf{B}}^{(\varrho+1-2\mu)}(r, \varphi) &= \sum_{n=\varrho-1-2\mu}^{\infty} \frac{\tilde{c}_n^{\varrho+1-2\mu}}{r^{n+2}} \mathbf{D}^{(n)}(\varphi), & \tilde{c}_{\varrho-1-2\mu}^{\varrho+1-2\mu} &\neq 0, \end{aligned} \right\} \mu = 1, \dots, (\varrho + 1 - \delta)/2$$

into the linear combination

$$\lambda_0 \mathbf{B}^{(\varrho+1)} + \sum_{\mu=1}^{(\varrho+1-\delta)/2} (\lambda_\mu \mathbf{B}^{(\varrho+1-2\mu)} + \tilde{\lambda}_\mu \tilde{\mathbf{B}}^{(\varrho+1-2\mu)}) = 0$$

and comparing successively the powers  $r^{-\delta}, \dots, r^{-(\varrho+1)}$  yields

$$\lambda_{(\varrho+1-\delta)/2} = \tilde{\lambda}_{(\varrho+1-\delta)/2} = 0, \quad \dots, \quad \lambda_1 = \tilde{\lambda}_1 = 0, \quad \lambda_0 = 0.$$

If  $\varrho + 1 - \delta$  is odd the  $\varrho - \delta + 1$  functions  $\mathbf{B}^{(\varrho+1)}$ ,  $\mathbf{B}^{(\varrho+1-2\mu)}$ , and  $\tilde{\mathbf{B}}^{(\varrho+1-2\mu)}$  with  $\mu = 1, \dots, (\varrho - \delta)/2$  have decay order  $\delta$  and are linear independent as before. Another linear independent solution is obtained by choosing a linear combination  $\hat{\mathbf{B}}$  of  $\mathbf{B}^{(\delta-1)}$  and  $\tilde{\mathbf{B}}^{(\delta-1)}$  such that the coefficient of  $r^{-(\delta-1)}$  vanishes. Since  $\mathbf{B}^{(\delta-1)}$  and  $\tilde{\mathbf{B}}^{(\delta-1)}$  are asymptotically independent  $\hat{\mathbf{B}}$  has now exact decay order  $\delta$  and is hence linear independent of all the other functions which have in fact decay orders  $\geq \delta + 1$ .  $\square$

REMARK: In the case of a bounded region  $A$  bounded by circles  $c_1$  and  $c_R$  with radii 1 and  $R$ , respectively, and direction fields  $\mathbf{D}$  and  $\hat{\mathbf{D}}$  with rotation numbers  $\varrho$  and  $\hat{\varrho} \leq \varrho$ , respectively, a similar reasoning as above (with the same prerequisites) yields for the dimension of the solution space  $L_{3a}(\mathbf{D}, \hat{\mathbf{D}})$ :

$$\dim L_{3a}(\mathbf{D}, \hat{\mathbf{D}}) = \varrho - \hat{\varrho} + 1. \quad (3.66)$$

According to Corollary 3.1 formula (3.66) holds in particular if the smallness conditions of Theorem 3.2 can be met.

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## References

Adams, A.A.: *Sobolev Spaces*. Academic Press, New York 1975.

- Ahlfors, L.V.: *Complex Analysis*. McGraw-Hill, Singapore 1979.
- Behnke, H., Sommer, F.: *Theorie der analytischen Funktionen einer komplexen Veränderlichen*. Springer-Verlag, Berlin, Heidelberg, New York 3rd edition 1972.
- Bers, L., John, F., Schechter, M.: *Partial Differential Equations*. Interscience Publishers, New York 1964.
- Folland, G.B.: *Partial Differential Equations*. Princeton University Press, Princeton, N.Y. 1995.
- Galdi, G.P.: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations Vol. I. Linearized Steady Problems*. Springer Tracts in Natural Philosophy Vol. 38, Springer-Verlag, Berlin, Heidelberg, New York 1994.
- Gilbarg, D., Trudinger, N. S.: *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, Heidelberg, New York 2nd edition 1998.
- Heinz, E.: *An elementary analytic theory of the degree of mapping in  $n$ -dimensional space*. Journal of Mathematics and Mechanics **8**, 231–247 (1959).
- Hulot, G., Khokhlov, A., Le Mouél, J.L.: *Uniqueness of mainly dipolar magnetic fields recovered from directional data*. Geophys. J. Int. **129**, 347–354 (1997).
- Hurwitz, A., Courant, R.: *Allgemeine Funktionentheorie und elliptische Funktionen. Geometrische Funktionentheorie*. Springer, Berlin 2nd edition 1925.
- Kaiser, R., Neudert, M.: *A non-standard boundary value problem related to geomagnetism*. Quarterly of Applied Mathematics **62**, 423–457 (2004).
- Kono, M.: *Uniqueness problems in the spherical harmonic analysis of the geomagnetic field direction data*. J. Geomag. Geoelectr. **28**, 11–29 (1976).
- Leis, R.: *Vorlesungen über partielle Differentialgleichungen zweiter Ordnung*. Bibliographisches Institut Mannheim, 1967.
- Martensen, E.: *Potentialtheorie*. Teubner-Verlag, Stuttgart 1968.
- Merrill, R.T., McElhinny, M.W.: *The Earth's Magnetic Field (Its History, Origin and Planetary Perspective)*. Academic Press, London 1983.
- Muskhelishvili, N.I.: *Singular Integral Equations*. Noordhof International Publishing, Leyden 1977.
- Proctor, M.R.E., Gubbins, D.: *Analysis of geomagnetic directional data*. Geophys. J. Int. **100**, 69–77 (1990).
- Simader, C.G.: *On Dirichlet's Boundary Value Problem*. Lecture Notes in Mathematics Vol. **268**. Springer-Verlag, Berlin, Heidelberg, New York 1972.
- Simader, C.G., Sohr, H.: *The Dirichlet problem for the Laplacian in bounded and unbounded domains*. Pitman Research Notes in Mathematics Series **360**. Addison Wesley Longman, Harlow 1996.