On the geomagnetic directional problem: A uniqueness result

Ralf Kaiser

Fakultät für Mathematik und Physik
Universität Bayreuth
D-95440 Bayreuth, Germany

Abstract

We consider the following nonlinear boundary value problem in the exterior space $\hat{V} = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| > 1\}$ of the unit sphere S: Given a vector field $\mathbf{D} : S \to \mathbb{R}^3$ we ask for all harmonic vector fields $\mathbf{B} : \hat{V} \to \mathbb{R}^3$ which decay at least as fast as a dipole field at infinity and are parallel to $\mathbf{D}$ on S, i.e. there is $f : S \to \mathbb{R}$ such that $\mathbf{B} = f\,\mathbf{D}$. This problem is related to the problem of reconstructing the geomagnetic field outside the earth from directional data measured on the earth's surface. We prove in this note the unique solvability of the above problem for a certain class of vectorfields, viz. all those vector fields $\mathbf{D}_n^k$, $n \in \mathbb{N}$, $0 < |k| < n$, which are obtained by restricting a single *nonaxisymmetric* multipole field on S.

Key words: Nonlinear boundary value problem, geomagnetism, directional problem
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1 Introduction

The problem to determine the magnetic field outside the earth if only the direction of the field is known on the earth's surface is well-known in geomagnetism. It is especially relevant for the interpretation of historical or palaeomagnetic data sets which provide directional information only.

Neglecting deviations from the spherical shape S of the earth's surface and assuming the exterior region $\hat{V}$ to be insulating and to be free of sources of magnetic field this problem can be formalized as follows: Given a direction field $\mathbf{D} \in C^0(S, \mathbb{R}^3)$ we ask for all nontrivial vector fields $\mathbf{B} \in C^1(\hat{V}, \mathbb{R}^3)$ for which a scalar function $f : S \to \mathbb{R}$ exists such that the conditions

$$\begin{aligned} \nabla \times \mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} = 0 \qquad & \text{in } \hat{V}, \\ |\mathbf{B}(\mathbf{x})| = O(|\mathbf{x}|^{-3}) \qquad & \text{for } |\mathbf{x}| \to \infty, \\ \mathbf{B} = f\,\mathbf{D} \qquad & \text{on } S \end{aligned} \tag{1}$$

are satisfied. It is the nonlinear boundary condition $(1)_3$ which makes the problem different from the standard boundary value problems of potential theory, which specify either the normal component or the tangential components on the boundary. The type of function f which is appropriate here and the precise sense in which $(1)_3$ holds is specified in the next section. Let us just remark that f need not obey a sign condition ("unsigned directional problem"), which makes the set of solutions $\mathbf{B}$ for *fixed* direction field $\mathbf{D}$ a linear space $L_{\mathbf{D}}$. Having in mind the reconstruction problem the dimension of this solution space, in particular, the question of uniqueness or nonuniqueness, is of major interest. Uniqueness is, of course, always understood up to a multiplicative constant which remains free.

An early attempt to prove uniqueness for arbitrary direction fields went astray (Kono 1976). This became obvious as Proctor & Gubbins (1990) gave the example of an axisymmetric direction field, for which they found numerically three different solutions. On the other side, Hulot et al. (1997) derived an upper bound on the dimension of the solution space, $\dim L_{\mathbf{D}} \le l_{\mathbf{D}} - 1$ with $l_{\mathbf{D}}$ being the number of "poles" of the direction field $\mathbf{D}$.

Kaiser & Neudert (2004) investigated the case of axisymmetric multipole fields: Neglecting the boundary condition the general solution of $(1)_1$, $(1)_2$ in spherical coordinates (r, θ, φ) allows the representation

$$\mathbf{B} = \nabla \Psi, \qquad \Psi = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} c_{nk} \, \frac{1}{r^{n+1}} \, Y_n^k(\theta, \varphi) \tag{2}$$

with Y_n^k being the usual spherical harmonics and c_{nk} arbitrary constants. Defining the basic direction fields or multipole fields

$$\mathbf{D}_n^k(\theta, \varphi) := \nabla \Big(\frac{1}{r^{n+1}} \, Y_n^k(\theta, \varphi) \Big) \Big|_{r=1}, \quad n \in \mathbb{N}, \quad k \in \{-n, \dots, n\} \tag{3}$$

there is obviously $\dim L_{\mathbf{D}_n^k} \ge 1$. In the case of axisymmetry it turned out that $L_{\mathbf{D}_n^0} = n$, $n \in \mathbb{N}$, which is precisely the upper bound in (Hulot et al. 1997). In the present note it is proved that for all other multipole fields there is uniqueness, i.e. $L_{\mathbf{D}_n^k} = 1$, $n \in \mathbb{N}$, $0 < |k| \le n$. The result is based on a

Hilbert space formulation of the problem, which is presented in the following section; section 3 contains the proof and some concluding remarks.

2 Hilbert space criterion

Let us introduce the Hilbert space

$$L^2(S) = \Big\{h : S \to \mathbb{C} \Big| \int\limits_S |h|^2 d\Omega < \infty\Big\}$$

with scalar product $(h_1, h_2) := \int_S h_1 h_2^* \, d\Omega$ and norm $\|h\| := (h, h)^{1/2}$. " $*$ " means complex conjugation. Boundary values are then explained in the sense of traces. Introducing, furthermore, a potential Ψ for the harmonic field $\mathbf{B}$ and using spherical coordinates (r, θ, φ) as well as the notation $\mathbf{x} = r\,\hat{\mathbf{x}}$ with unit vector $\hat{\mathbf{x}}$ the problem (1) takes the following form:

Problem: *Let* $\mathbf{D} = D_r(\theta, \varphi)\mathbf{e}_r + D_\theta(\theta, \varphi)\mathbf{e}_\theta + D_\varphi(\theta, \varphi)\mathbf{e}_\varphi \in C^0(S, \mathbb{C}^3)$. *Determine all functions* $f : S \to \mathbb{C}$ *such that* $f\,\mathbf{D} \in L^2(S)^3$ *and that the following boundary value problem is solvable, i.e. there is* $\Psi \in C^2(\hat{V}, \mathbb{C})$ *satisfying the conditions:*

$$\begin{aligned} \Delta\Psi &= 0 && \text{in } \hat{V}, \\ |\Psi(\mathbf{x})| &= O(|\mathbf{x}|^{-2}) && \text{for } |\mathbf{x}| \to \infty, \\ \nabla\Psi &= f\,\mathbf{D} && \text{on } S. \end{aligned} \tag{4}$$

The boundary condition $(4)_3$ *holds in the sense of*

$$\begin{aligned} \|\partial_r \Psi(r.)|_S - f\, D_r\| &\to 0 \quad \text{for } r \searrow 1, \\ \|\partial_\theta \Psi(r.)|_S \sin\theta - f\, D_\theta \sin\theta\| &\to 0 \quad \text{for } r \searrow 1, \\ \|\partial_\varphi \Psi(r.)|_S - f\, D_\varphi \sin\theta\| &\to 0 \quad \text{for } r \searrow 1. \end{aligned}$$

The formulation of the Problem is for technical reasons slightly more general than that in (Kaiser & Neudert 2004). In fact, only those solutions make physical sense, where at least one of the quantities f and $\mathbf{D}$ is real. Taking the real or imaginary part of the other quantity yields then a reasonable solution. We have now the following

Criterion: *Let* $\mathbf{D} \in C^0(S, \mathbb{C}^3)$ *as in the Problem and* $f : S \to \mathbb{C}$ *such that* $f\,\mathbf{D} \in L^2(S)^3$*. Moreover, let for* $n \in \mathbb{N}, k \in \{-n, ..., n\}$

$$\begin{aligned}
T_{-1} &:= D_r^*,\\
T_0^0 &:= D_\theta^* \sin\theta\, Y_0^0 - \tfrac{1}{\sqrt{3}}\, D_r^*\, Y_1^0,\\
T_n^k &:= D_\theta^* \sin\theta\; Y_n^k + D_r^* \Big(\tfrac{n-1}{n} \sqrt{\tfrac{n^2-k^2}{4n^2-1}}\, Y_{n-1}^k - \sqrt{\tfrac{(n+1)^2-k^2}{4(n+1)^2-1}}\, Y_{n+1}^k \Big),\\
S_0^0 &:= D_\varphi^* \sin\theta\, Y_0^0,\\
S_n^k &:= (-ik\, D_r^* + (n+1)\, D_\varphi^* \sin\theta)\; Y_n^k,
\end{aligned}$$

where $Y_n^k = Y_n^k(\theta, \varphi)$ *are spherical harmonics, and*

$$L_{\mathbf{D}} := \Big\{ f \Big| f\,\mathbf{D} \in L^2(S)^3 \text{ and } (f, T_{-1}) = (f, T_n^k) = (f, S_n^k) = 0,\;\; n \in \mathbb{N}_0,\;\; |k| \le n \Big\}.$$

Then f *is a solution of the Problem if and only if* $f \in L_{\mathbf{D}}$.

Again, the Criterion is a slightly more general version of the Theorem 2.4 in (Kaiser & Neudert 2004). The proof is analogous to that of Theorem 2.4 and need not be reproduced here.

3 Uniqueness for nonaxisymmetric multipole fields

The central result of the present note is the following

Theorem: *Let* $\mathbf{D} = \mathbf{D}_N^K$*,* $N \in \mathbb{N}$*,* $0 < |K| \le N$ *be a single nonaxisymmetric multipole field. We have then* $\dim L_{\mathbf{D}_N^K} = 1$*, i.e. the Problem is uniquely solvable. In particular,* $f = const = c$ *and* c *can be chosen to be real.*

PROOF: The proof is based on the Criterion. Evaluating definition (3) yields the spherical components of $\mathbf{D}_N^K$

$$\mathbf{D}_N^K = -(N+1) Y_N^K\, \mathbf{e}_r + \partial_\theta Y_N^K\, \mathbf{e}_\theta + \frac{i\,K}{\sin\theta}\, Y_N^K\, \mathbf{e}_\varphi. \tag{5}$$

Let now $f \in L_{\mathbf{D}_N^K}$. Since $\Big\{ Y_n^k \Big| n \in \mathbb{N}_0,\, k \in \{-n, \ldots, n\} \Big\}$ is a complete orthonormal system in $L^2(S)$ the condition $f\, D_r \in L^2(S)$ from the Criterion implies the representation

$$f = \frac{1}{Y_N^K} \sum_{n=0}^{\infty} \sum_{k=-n}^{n} c_{nk}\, Y_n^k. \tag{6}$$

Inserting (6) in the condition $(f, S_n^k) = 0$ yields

$$(-k(N+1) + (n+1)K)\, c_{nk} = 0, \quad n \in \mathbb{N}_0 \;\; |k| \le n. \tag{7}$$

Obviously the choice $c_{nk} := c\, \delta_{nN}\, \delta_{kK}$ with $c \in \mathbb{C}$ satisfies (7) and, moreover, $f_c := c$ satisfies also $f_c\, \mathbf{D}_\theta \in L^2(S)$, $f_c\, \mathbf{D}_\varphi \in L^2(S)$ as well as $(f_c, T_{-1}) = (f_c, T_n^k) = 0$. Thus, $f_c \in L_{\mathbf{D}_N^K}$

We show next that there are no further solutions. It follows from (7) that $c_{nk} \neq 0$ only if $k = \frac{K}{N+1}(n+1)$. So, for fixed $n \in \mathbb{N}_0$ there is at most one $k \in \mathbb{Z}$ such that $c_{nk} \neq 0$. Let (n_l, k_l) denote all these pairs except (N, K) with l running through $\{1, \ldots, L\}$ or $\mathbb{N}$. The representation (6) takes now the form

$$f = \frac{1}{Y_N^K} \sum_l c_{n_l k_l}\, Y_{n_l}^{k_l}, \tag{8}$$

and we conclude from the condition $f\, \mathbf{D}_\theta \in L^2(S)$:

$$\begin{aligned} \infty &> \Big\| \frac{\partial_\theta Y_N^K}{Y_N^K} \sum_l c_{n_l k_l} Y_{n_l}^{k_l} \Big\|^2 \\ &= \int_0^{2\pi}\!\!\int_0^{\pi} \Big| \frac{\partial_\theta Y_N^K(\theta,\varphi)}{Y_N^K(\theta,\varphi)} \sum_l c_{n_l k_l} Y_{n_l}^{k_l}(\theta,\varphi) \Big|^2 \sin\theta \, d\theta d\varphi \\ &= \sum_l |c_{n_l k_l}|^2 \Big\| \frac{\partial_\theta Y_N^K}{Y_N^K}\, Y_{n_l}^{k_l} \Big\|^2 . \end{aligned} \tag{9}$$

Here, we made use of the representation (cf. Abramowitz & Stegun 1972)

$$Y_n^k(\theta,\varphi) = d_{nk}\, P_n^{|k|}(\cos\theta)\, e^{ik\varphi},$$

where $P_n^{|k|}(\cdot)$ are associated Legendre polynomials and d_{nk} are some constants $\neq 0$. On the other side, we have for $|K| \neq N$

$$\Big\| \frac{\partial_\theta Y_N^K}{Y_N^K}\, Y_{n_l}^{k_l} \Big\|^2 = 2\pi\, |d_{n_l k_l}|^2 \int_{-1}^{1} \frac{|P_N'^{|K|}(x)|^2}{|P_N^{|K|}(x)|^2}\, |P_{n_l}^{|k_l|}(x)|^2 \sqrt{1-x^2}\, dx = \infty,$$

which follows from the fact that $P_N^{|K|}(\cdot)$ has for $|K| \neq N$ always a zero in the interval $(-1,1)$ which is not balanced by zeros of the other terms and thus generates a nonintegrable singularity. This is a contradiction to (9), hence all $c_{n_l k_l}$ must vanish. In the case $|K| = N$ we can write $\sin\theta\, \partial_\theta\, Y_N^K(\theta,\varphi) =$

$N \cos\theta\, Y_N^K(\theta,\varphi)$, and the condition $(f, T_n^k) = 0$ amounts to

$$(2N+1)\sqrt{\tfrac{(n+1)^2-k^2}{4(n+1)^2-1}}\, c_{n+1\,k} = \left(1 - \tfrac{N+1}{n}\right)\sqrt{\tfrac{n^2-k^2}{4n^2-1}}\, c_{n-1\,k}, \quad n \in \mathbb{N}, \quad |k| \le n. \quad (10)$$

From (7) and (10) one concludes again that all $c_{nk} \neq c_{NK}$ must vanish. □

We conclude with some remarks:

1. Computing the number of poles for the nonaxisymmetric multipole fields $\mathbf{D}_n^k$, $n \in \mathbb{N}$, $0 < |k| \le n$ we find $l_{\mathbf{D}_n^k} = 4k(n-k) + 2(k+1-\delta_{1k})$. So, in that case the upper bound on the dimension of the solution space, $\dim L_{\mathbf{D}} \le l_{\mathbf{D}} - 1$, as given in (Hulot et al. 1997), largely overestimates the correct value $\dim L_{\mathbf{D}} = 1$.

2. Our result suggests uniqueness in the directional problem for *all* nonaxisymmetric direction fields. This conjecture is supported by an analysis of the direction field $\mathbf{D}_\lambda := \mathbf{D}_2^0 + \lambda\,\mathbf{D}_1^1$, where we found uniqueness for arbitrarily small nonzero values of λ. Proctor & Gubbins (1990) tested a number of finite-dimensional field models with respect to uniqueness. They used a numerical procedure which is inspired by linear inverse theory. The method is not rigorous; however, it suggested likewise uniqueness in all nonaxisymmetric models.

3. The problem of the existence of solutions is trivial for the multipole fields $\mathbf{D}_n^k$, but it is not trivial in the general case. For example, there are no solutions at all of the Problem for the direction fields $\tilde{\mathbf{D}}_n^k(\theta,\varphi) := \nabla(r^n\, Y_n^k(\theta,\varphi))|_{r=1}$.

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