

On purely toroidal dynamo magnetic fields caused by conductivity variations

R. Kaiser*

Fakultät für Mathematik, Physik und Informatik
Universität Bayreuth
D-95440 Bayreuth, Germany

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Abstract

Solutions of the induction equation with a purely toroidal magnetic field in a spherical conductor are of special interest since they disclose the possibility of a dynamo operating in a celestial body which remains invisible to an external observer. A necessary condition for such solutions to exist are flow fields and conductivity distributions constrained in such a way that no poloidal field can arise. Otherwise these quantities are assumed to be arbitrary; we allow, in particular, non-radial conductivity variations in order to facilitate dynamo action.

This paper presents evidence that such solutions do not exist: Based on a solution of the constraint a nonlinear evolution equation for the toroidal scalar T is derived. It is proved that sufficiently regular solutions of this equation cannot increase (in time) in the norm $\int_0^R (\max_{|\mathbf{r}|=r} T - \min_{|\mathbf{r}|=r} T) r dr$, where R is the radius of the spherical conductor. Steady solutions are likewise excluded.

Key Words: Dynamo theory, antidynamo theorem.

1 Introduction

In dynamo theory the decomposition of the magnetic field in poloidal and toroidal components is an elegant and popular method to eliminate the divergence constraint. In particular, if the conductor is a ball the toroidal component is confined to this ball. So, a dynamo solution of the induction equation consisting only of the toroidal component would leave no trace in the vacuum region surrounding the conductor, and a dynamo of this type operating in a celestial body would be invisible for an external observer.

However, the existence of such invisible dynamos has been doubted early on (cf. Bullard and Gellman 1954, Elsasser 1956, Childress 1969, Busse 1977, Ivers and James 1988), but no pertinent antidynamo theorem could be derived. The first hard piece of evidence for such a theorem appeared in (Kaiser *et al.* 1994): With the ansatz of a purely toroidal magnetic field in the induction equation an evolution equation for the toroidal scalar T has been derived

*Email: ralf.kaiser@uni-bayreuth.de

together with a constraint equation involving the non-radial derivatives of T and of the radial component v_r of the flow field. This constraint is not automatically preserved by the evolution equation and has to be solved in some way. Incorporating such a solution of the constraint into the evolution equation led to a nonlinear parabolic equation for T with vanishing zeroth-order term and first-order terms which have neither pure advection form nor pure divergence form. It was this mixed form of the first-order terms, which prevented the straightforward application of maximum principles or L^2 - (i.e. energy) or L^1 - estimates, which are the basic mathematical tools in deriving the well-known classical antidynamo theorems (cf. Ivers and James 1984, Backus 1958, Kaiser 2007). Instead, employing an “extremal-path” argument monotonous decay of T in a mixed L^1 - L^∞ -norm has been shown for solutions satisfying a certain regularity condition. This result has recently been extended and improved (Kaiser 2009): The regularity condition turned out to be redundant, the mixed norm could be replaced by the uniform maximum norm, and the decay was in fact (exponentially fast) to zero.

So far an implied assumption of the above results is spherical symmetry of the conductivity distribution λ . However, as demonstrated in plane geometry, lateral conductivity variations can assist a flow field at magnetic field generation which is otherwise not capable of dynamo action (Busse and Wicht 1992): Dynamo action has been found in a horizontally unbounded layer for a flow field without perpendicular component, a situation where in the case of a horizontally uniform λ dynamo action is excluded by the toroidal velocity theorem. The present paper explores the possibility of a dynamo generating a purely toroidal magnetic field assisted by non-radial conductivity variations and presents evidence that no such dynamo exists.

Our procedure is analogous to that in (Kaiser *et al.* 1994): We derive an evolution equation for T and a constraint equation involving non-radial derivatives of T , v_r , and λ . Based on a solution of the constraint we apply the extremal-path method to the resulting nonlinear evolution equation and find monotonous decay of the quantity $\int_0^R (\max_{|\mathbf{r}|=r} T - \min_{|\mathbf{r}|=r} T) r dr$ in time, if T is sufficiently regular. The possibility of steady solutions which are not yet ruled out by the decay result can be eliminated by a similar argument.

Plan of the paper: Section 2 gives a precise statement of the dynamo problem, derives the constraint equation, presents a solution thereof, and formulates the resulting nonlinear evolution equation for the (scaled) toroidal scalar \mathcal{T} . Section 3 formulates a regularity condition for \mathcal{T} and proves the decay result whereas section 4 deals with the steady case. Section 5, finally, contains the conclusions and some hints at desirable improvements. Some useful relations concerning radial and non-radial derivatives in spherical geometry are collected in an appendix.

2 An evolution equation for the toroidal scalar

Starting point is the kinematic dynamo problem

$$\left. \begin{aligned} \partial_t \mathbf{B} &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\lambda \nabla \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0 && \text{in } B_R \times (0, \infty), \\ \nabla \times \mathbf{B} &= 0, \quad \nabla \cdot \mathbf{B} = 0 && \text{in } \mathbb{R}^3 \setminus \overline{B_R} \times (0, \infty), \\ \mathbf{B} &\text{ continuous} && \text{in } \mathbb{R}^3 \times (0, \infty), \\ |\mathbf{B}(\mathbf{r}, \cdot)| &\rightarrow 0 && \text{for } |\mathbf{r}| \rightarrow \infty, \\ \mathbf{B}(\cdot, 0) &= \mathbf{B}_0, \quad \nabla \cdot \mathbf{B}_0 = 0 && \text{on } B_R \times \{t = 0\}, \end{aligned} \right\} \quad (1)$$

where the conducting region B_R is a ball of radius R (cf. e.g. Moffatt 1978). The flow field \mathbf{v} and the conductivity $\lambda > 0$ are prescribed quantities, which, for the present, are assumed to be arbitrary smooth functions of space and time. In particular, \mathbf{v} need not be solenoidal and λ need not be spherically symmetric.

The basic assumption is now that of a purely toroidal magnetic field, i.e. \mathbf{B} has the form (cf. Backus 1958): ¹

$$\mathbf{B} = \nabla T \times \mathbf{r} = -(\mathbf{r} \times \nabla)T =: -\Lambda T \quad (2)$$

with some scalar function T , which can assumed to have vanishing mean over spheres S_r of radius r ,

$$\langle T \rangle := (4\pi r^2)^{-1} \int_{S_r} T ds = 0 \quad \text{for } r \in [0, R]. \quad (3)$$

A magnetic field of the form (2) has obviously no radial component. Since there are no nontrivial harmonic fields in $\mathbb{R}^3 \setminus \overline{B_R}$ with vanishing normal component on S_R and satisfying (1)₄, magnetic fields of this type are in fact confined to B_R and vanish (because of (1)₃) on S_R . In B_R the induction equation (1)₁ can equivalently be written by taking the radial components of the equation and of its curl (cf. Backus 1958). Inserting the representation (2) one obtains:

$$0 = \mathbf{r} \cdot \nabla \times (\mathbf{v} \times \Lambda T) - \mathbf{r} \cdot \nabla \times (\lambda \nabla \times \Lambda T), \quad (4)$$

$$\mathbf{r} \cdot \nabla \times \partial_t \Lambda T = \mathbf{r} \cdot \nabla \times [\nabla \times (\mathbf{v} \times \Lambda T) - \nabla \times (\lambda \nabla \times \Lambda T)]. \quad (5)$$

To evaluate these equations it is convenient to introduce derivative-operators adapted to spherical geometry. We define radial and non-radial derivatives

$$\partial_r := (\mathbf{r}/r) \cdot \nabla, \quad \nabla_{nr} := \nabla - (\mathbf{r}/r) \partial_r \quad (6)$$

and find

$$\Lambda = \mathbf{r} \times \nabla = \mathbf{r} \times \nabla_{nr}, \quad \nabla_{nr} = -\frac{1}{r^2} (\mathbf{r} \times \Lambda). \quad (7)$$

The Laplace-Beltrami operator \mathcal{L} on the unit sphere then takes the form

$$\mathcal{L} = \Lambda \cdot \Lambda = r^2 \nabla_{nr} \cdot \nabla_{nr}; \quad (8)$$

other useful relations between these operators can be found in the appendix. Note, in particular, that ∂_r interchanges with Λ and with $r \nabla_{nr}$, but Λ and ∇_{nr} do not. In the following we will make free use of the pertinent relations (A2) and (A3).

Equation (5) takes now with (7) and (8) the form

$$\mathcal{L} \partial_t T = \Lambda \cdot \nabla \times (\mathbf{v} \times \Lambda T) - \Lambda \cdot \nabla \times (\lambda \nabla \times \Lambda T). \quad (9)$$

To rewrite the first term on the right-hand side of (9) we make use of the decomposition

$$\mathbf{v} \times \Lambda T = \mathbf{v} \times (\mathbf{r} \times \nabla T) = \mathbf{r}(\mathbf{v} \cdot \nabla_{nr} + v_r \partial_r)T - r v_r \nabla T. \quad (10)$$

To compute the curl of the last term on the right-hand side we split the gradients in radial and non-radial parts and find with (7) and (A5)

$$\nabla \times (r v_r \nabla T) = \frac{1}{r} \Lambda v_r \times \Lambda T + \frac{v_r}{r} \Lambda T + \partial_r v_r \Lambda T - \Lambda v_r \partial_r T. \quad (11)$$

¹Note that the representation (2) differs from that used by Kaiser *et al.* (1994) or Backus (1958) by a sign.

Thus, with (8) the first term takes the form

$$\begin{aligned}\Lambda \cdot \nabla \times (\mathbf{v} \times \Lambda T) &= \Lambda \cdot \left[-\frac{1}{r} \Lambda v_r \times \Lambda T - \Lambda (\mathbf{v} \cdot \nabla_{nr} T) - \frac{1}{r} \partial_r (r v_r \Lambda T) \right] \\ &= -\mathcal{L}(\mathbf{v} \cdot \nabla_{nr} T) - \frac{1}{r} \Lambda \cdot \partial_r (v_r \Lambda (rT)).\end{aligned}\quad (12)$$

Note that the first term in square brackets in (12) is purely radial and consequently does not contribute. To rewrite the second term on the right-hand side of (9) we start with curl \mathbf{B} . Splitting again the gradient we find with (A8)

$$\nabla \times \Lambda T = (\mathbf{r}/r^2) \mathcal{L} T - \nabla_{nr} T - r \nabla_{nr} \partial_r T = (\mathbf{r}/r^2) \mathcal{L} T - \nabla_{nr} \partial_r (rT), \quad (13)$$

and therefore with (A5) and (A7)

$$\begin{aligned}\Lambda \cdot \nabla \times (\lambda \nabla \times \Lambda T) &= \Lambda \cdot \left[-\Lambda \cdot \left(\frac{1}{r^2} \lambda \mathcal{L} T \right) - \frac{1}{r^2} \Lambda \lambda \times \Lambda \partial_r (rT) \right. \\ &\quad \left. - \frac{1}{r^2} \lambda \Lambda \partial_r (rT) - \partial_r \left(\frac{1}{r} \lambda \Lambda \partial_r (rT) \right) \right] \\ &= -\mathcal{L} \left(\frac{1}{r^2} \lambda \mathcal{L} T \right) - \frac{1}{r} \Lambda \cdot \partial_r (\lambda \Lambda \partial_r (rT)).\end{aligned}\quad (14)$$

Introducing the scaled toroidal scalar $\mathcal{T} := rT$ we can thus rewrite eq. (9) as

$$\mathcal{L} \partial_t \mathcal{T} = \mathcal{L} \left(\frac{\lambda}{r^2} \mathcal{L} T \right) + \Lambda \cdot \partial_r (\lambda \Lambda \partial_r \mathcal{T}) - \mathcal{L}(\mathbf{v} \cdot \nabla_{nr} T) - \Lambda \cdot \partial_r (v_r \Lambda T). \quad (15)$$

On the other side, with (7), (10), (13), and (A4) the constraint (4) takes the form

$$0 = \Lambda \cdot (r v_r \nabla T) - \Lambda \cdot (\lambda \nabla_{nr} \partial_r (rT)) \quad (16)$$

or, with (A6) and expressed in terms of \mathcal{T} :

$$0 = \Lambda v_r \cdot \nabla_{nr} \mathcal{T} - \Lambda \lambda \cdot \nabla_{nr} \partial_r \mathcal{T}. \quad (17)$$

Equation (17) can now be solved by introducing functions $w(\tau, r, t)$ and $\mu(\tau, r, t)$ such that

$$v_r(\mathbf{r}, t) = w(\mathcal{T}(\mathbf{r}, t), r, t), \quad \lambda(\mathbf{r}, t) = \mu(\partial_r \mathcal{T}(\mathbf{r}, t), r, t). \quad (18)$$

Moreover, substituting (18) in (15), a second-order parabolic equation for \mathcal{T} can be derived from (15). In fact, with

$$v_r \Lambda T = \Lambda \int_0^{\mathcal{T}} w(\tau, \cdot, \cdot) d\tau =: \Lambda W(\mathcal{T}, \cdot, \cdot), \quad (19)$$

$$\lambda \Lambda \partial_r T = \Lambda \int_0^{\partial_r \mathcal{T}} \mu(\tau, \cdot, \cdot) d\tau =: \Lambda M(\partial_r \mathcal{T}, \cdot, \cdot) \quad (20)$$

eq. (15) takes the form

$$\mathcal{L} \partial_t \mathcal{T} = \mathcal{L} \left[\frac{\lambda}{r^2} \mathcal{L} T + \partial_r M(\partial_r \mathcal{T}, \cdot, \cdot) - \mathbf{v} \cdot \nabla_{nr} \mathcal{T} - \partial_r W(\mathcal{T}, \cdot, \cdot) \right]. \quad (21)$$

Therefore, after eliminating \mathcal{L} we are left with the following initial-boundary-value problem in $B_R \times (0, \infty)$:

$$\left. \begin{aligned} \partial_t \mathcal{T} &= \frac{\lambda}{r^2} \mathcal{L}\mathcal{T} + \partial_r M(\partial_r \mathcal{T}, \cdot, \cdot) - \mathbf{v} \cdot \nabla_{nr} \mathcal{T} \\ &\quad - \partial_r W(\mathcal{T}, \cdot, \cdot) - sm && \text{in } B_R \times (0, \infty), \\ \mathcal{T} &= 0 && \text{on } (S_R \cup \{r = 0\}) \times (0, \infty), \\ \mathcal{T}(\cdot, 0) &= \mathcal{T}_0, \quad \langle \mathcal{T}_0 \rangle = 0 && \text{on } B_R \times \{t = 0\}, \end{aligned} \right\} \quad (22)$$

where $sm = sm(r, t)$ is the spherical mean

$$sm := \left\langle \frac{\lambda}{r^2} \mathcal{L}\mathcal{T} + \partial_r M(\partial_r \mathcal{T}, \cdot, \cdot) - \mathbf{v} \cdot \nabla_{nr} \mathcal{T} - \partial_r W(\mathcal{T}, \cdot, \cdot) \right\rangle. \quad (23)$$

Remarks: 1. It is not guaranteed that any solution of the constraint has the form (18). Only in the special cases $\nabla_{nr} \lambda \equiv 0$ or $\nabla_{nr} v_r \equiv 0$ we have further evidence for solutions of this type. In the former case any solution is locally of the form (18)₁ if $\Lambda T = -\mathbf{B} \neq 0$ and in the latter case solutions are locally of the form (18)₂ if $\partial_r \mathbf{B} \neq 0$.

2. Equation (22)₁ has not quite the standard form of a (nonlinear) second-order parabolic equation. It differs in the nonlocal term sm and the (in the origin) singular coefficients. The subsequent treatment of the equation has to take these difficulties into consideration.

3 Monotonous decay of the toroidal scalar

In this section monotonous decay in a mixed L^1 - L^∞ -norm of smooth solutions of problem (22) satisfying, additionally, a certain regularity condition (see below) is proved. The mixed norm reflects the different character radial and non-radial derivatives play in eq. (22)₁. “Smooth” means here that \mathcal{T} , $\partial_t \mathcal{T}$, and all spatial derivatives of \mathcal{T} up to the second order are continuous functions on the space-time cylinder $B_R \times (0, t^*)$ for any $t^* > 0$; \mathcal{T} and $\nabla \mathcal{T}$ are, moreover, continuous on the closed cylinder $\overline{B_R} \times [0, t^*]$. All other quantities are assumed to be sufficiently smooth to ensure such a solution \mathcal{T} ; in particular, W , $\partial_r W$, $\partial_\tau W$ as well as M , $\partial_r M$, $\partial_\tau M$ are continuous functions on their respective domains. The proof is based on the “extremal-path” argument as already used by Kaiser *et al.* (1994). Let $\mathbf{r} = r\hat{\mathbf{r}}$ with $\hat{\mathbf{r}}$ varying on the unit sphere S_1 and consider maxima and minima of \mathcal{T} on spheres S_r with radius r :

$$\left. \begin{aligned} \mathcal{T}_{max}(r, t) &:= \max_{\mathbf{r} \in S_r} \mathcal{T}(\mathbf{r}, t) = \max_{\hat{\mathbf{r}} \in S_1} \mathcal{T}(r\hat{\mathbf{r}}, t), \\ \mathcal{T}_{min}(r, t) &:= \min_{\mathbf{r} \in S_r} \mathcal{T}(\mathbf{r}, t) = \min_{\hat{\mathbf{r}} \in S_1} \mathcal{T}(r\hat{\mathbf{r}}, t). \end{aligned} \right\} \quad (24)$$

Due to the zero-mean condition (3) we have $\mathcal{T}_{max} \geq 0$ and $\mathcal{T}_{min} \leq 0$. Let $\mathbf{r}_m(r, t)$, $m \in \{max, min\}$ denote the loci on S_1 where these extrema are attained, i.e.

$$\mathcal{T}_m(r, t) = \mathcal{T}(r\hat{\mathbf{r}}_m(r, t), t). \quad (25)$$

For fixed t the function $r \mapsto r\hat{\mathbf{r}}_m(r, t)$ can be viewed as “path” in B_R connecting the extrema on different spheres. This path need not be unique and, whereas $\mathcal{T}_m(r, t)$ is a continuous function, $\hat{\mathbf{r}}_m(r, t)$ need not be continuous either (cf. Kaiser *et al.* 1994). In order that the

functions $\hat{\mathbf{r}}_m(r, t)$ are not too “wild” we impose the following regularity condition on \mathcal{T} :

$$\begin{aligned} & \text{There is a finite partition } \{t_i\}_{i \leq i_0} \text{ of } [0, t^*] \text{ and a (in general time-dependent)} \\ & \text{partition } \{r_j\}_{j \leq j_0(i)} \text{ of } [0, R] \text{ such that } r_j \text{ and } (d/dt)r_j \text{ are continuous on } [t_{i-1}, t_i] \\ & \text{and } \hat{\mathbf{r}}_m, \partial_t \hat{\mathbf{r}}_m, \partial_r \hat{\mathbf{r}}_m, \text{ and } \partial_r^2 \hat{\mathbf{r}}_m, m \in \{max, min\} \text{ are continuous on } \overline{D_{ij}} \text{ with} \\ & D_{ij} := \{(r, t) \in (r_{j-1}(t), r_j(t)) \times (t_{i-1}, t_i)\} \text{ for } 1 \leq j \leq j_0(i), 1 \leq i \leq i_0. \end{aligned} \quad (26)$$

Condition (26) states essentially that up to finitely many exceptional times t_i the functions $r \mapsto r\hat{\mathbf{r}}_m(r, t)$ for fixed t are smooth paths with at most finitely many “jumps”. Regularity conditions of this type are always necessary if calculations involving extremal paths are performed. They are implicitly assumed in earlier work applying such arguments as, for example, in (Hide and Palmer 1982).

We evaluate now eq. (22)₁ at the extremal loci $\hat{\mathbf{r}}_m(r, t)$ to find “reduced” differential inequalities on $(0, R) \times (0, t^*)$. For that purpose we make use of the following (in-)equalities, whose proofs can be found in (Kaiser *et al.* 1994, Appendix A):

$$\partial_r \mathcal{T}(r\hat{\mathbf{r}}_m, t) = \partial_r \mathcal{T}_m(r, t), \quad \partial_t \mathcal{T}(r\hat{\mathbf{r}}_m, t) = \partial_t \mathcal{T}_m(r, t), \quad \nabla_{nr} \mathcal{T}(r\hat{\mathbf{r}}_m, t) = 0, \quad (27)$$

$$\mathcal{L}\mathcal{T}(r\hat{\mathbf{r}}_{max}, t) \leq 0, \quad \mathcal{L}\mathcal{T}(r\hat{\mathbf{r}}_{min}, t) \geq 0, \quad (28)$$

$$\partial_r^2 \mathcal{T}(r\hat{\mathbf{r}}_{max}, t) \leq \partial_r^2 \mathcal{T}_{max}(r, t), \quad \partial_r^2 \mathcal{T}(r\hat{\mathbf{r}}_{min}, t) \geq \partial_r^2 \mathcal{T}_{min}(r, t). \quad (29)$$

From its definition (20) we have

$$\partial_r M(\partial_r \mathcal{T}, r, t) = \int_0^{\partial_r \mathcal{T}} \partial_r \mu(\tau, r, t) d\tau + \mu(\partial_r \mathcal{T}, r, t) \partial_r^2 \mathcal{T}. \quad (30)$$

Thus, noting that $\mu > 0$ (since $\lambda > 0$) and using (27) and (29) we can evaluate $\partial_r M$ at \mathcal{T}_{max} to find

$$\begin{aligned} \partial_r M(\partial_r \mathcal{T}, r, t)|_{\mathcal{T}=\mathcal{T}_{max}} & \leq \int_0^{\partial_r \mathcal{T}_{max}} \partial_r \mu(\tau, r, t) d\tau + \mu(\partial_r \mathcal{T}_{max}, r, t) \partial_r^2 \mathcal{T}_{max} \\ & = \partial_r M(\partial_r \mathcal{T}_{max}, r, t), \end{aligned} \quad (31)$$

and analogously:

$$\partial_r M(\partial_r \mathcal{T}, r, t)|_{\mathcal{T}=\mathcal{T}_{min}} \geq \partial_r M(\partial_r \mathcal{T}_{min}, r, t). \quad (32)$$

Finally, we have

$$\partial_r W(\mathcal{T}, r, t)|_{\mathcal{T}=\mathcal{T}_m} = \partial_r W(\mathcal{T}_m, r, t). \quad (33)$$

Inserting (27) - (33) into eq. (22)₁ then yields

$$\partial_t \mathcal{T}_{max} \leq \partial_r M(\partial_r \mathcal{T}_{max}, \cdot, \cdot) - \partial_r W(\mathcal{T}_{max}, \cdot, \cdot) - sm, \quad (34)$$

$$\partial_t \mathcal{T}_{min} \geq \partial_r M(\partial_r \mathcal{T}_{min}, \cdot, \cdot) - \partial_r W(\mathcal{T}_{min}, \cdot, \cdot) - sm. \quad (35)$$

According to the regularity condition (26) the eqs. (34), (35) hold in any region D_{ij} . Indicating the interval (r_{j-1}, r_j) by an index j , subtracting eq. (35) from (34), and integrating over r one obtains

$$\begin{aligned} \int_{r_{j-1}}^{r_j} \partial_t (\mathcal{T}_{max}^j - \mathcal{T}_{min}^j) dr & \leq [M(\partial_r \mathcal{T}_{max}^j, \cdot, \cdot) - M(\partial_r \mathcal{T}_{min}^j, \cdot, \cdot)]_{r_{j-1}}^{r_j} \\ & \quad - [W(\mathcal{T}_{max}^j, \cdot, \cdot) - W(\mathcal{T}_{min}^j, \cdot, \cdot)]_{r_{j-1}}^{r_j}. \end{aligned} \quad (36)$$

As in (Kaiser *et al.* 1994) at the “boundaries” of the interval (r_{j-1}, r_j) the following inequalities hold

$$\left. \begin{aligned} \partial_r \mathcal{T}_{max}^{j+1}(r_j, \cdot) &\geq \partial_r \mathcal{T}_{max}^j(r_j, \cdot) \\ \partial_r \mathcal{T}_{min}^{j+1}(r_j, \cdot) &\leq \partial_r \mathcal{T}_{min}^j(r_j, \cdot) \end{aligned} \right\}, \quad 1 \leq j \leq j_0 - 1, \quad (37)$$

and at $r_0 = 0$ and $r_{j_0} = R$ follows from (22)₂:

$$\left. \begin{aligned} \partial_r \mathcal{T}_{max}^1(0, \cdot) &\geq 0, & \partial_r \mathcal{T}_{max}^{j_0}(R, \cdot) &\leq 0, \\ \partial_r \mathcal{T}_{min}^1(0, \cdot) &\leq 0, & \partial_r \mathcal{T}_{min}^{j_0}(R, \cdot) &\geq 0. \end{aligned} \right\} \quad (38)$$

Thus, summing up over j one obtains for the first term on the right-hand side of (36):

$$\left. \begin{aligned} &\sum_{j=1}^{j_0} [M(\partial_r \mathcal{T}_{max}^j, \cdot, \cdot) - M(\partial_r \mathcal{T}_{min}^j, \cdot, \cdot)]_{r_{j-1}}^{r_j} = \\ &\quad -M(\partial_r \mathcal{T}_{max}^1(0, \cdot), 0, \cdot) + M(\partial_r \mathcal{T}_{min}^1(0, \cdot), 0, \cdot) \\ &\quad + \sum_{j=1}^{j_0-1} \{M(\partial_r \mathcal{T}_{max}^j(r_j, \cdot), r_j, \cdot) - M(\partial_r \mathcal{T}_{max}^{j+1}(r_j, \cdot), r_j, \cdot)\} \\ &\quad - \sum_{j=1}^{j_0-1} \{M(\partial_r \mathcal{T}_{min}^j(r_j, \cdot), r_j, \cdot) - M(\partial_r \mathcal{T}_{min}^{j+1}(r_j, \cdot), r_j, \cdot)\} \\ &\quad + M(\partial_r \mathcal{T}_{max}^{j_0}(R, \cdot), R, \cdot) - M(\partial_r \mathcal{T}_{min}^{j_0}(R, \cdot), R, \cdot) \leq 0, \end{aligned} \right\} \quad (39)$$

where we used the fact that $\tau \mapsto M(\tau, \cdot, \cdot)$ is a monotonically increasing function with $M(0, \cdot, \cdot) = 0$. The second term is simpler to evaluate. Merely with the continuity of \mathcal{T}_m and W , with $W(0, \cdot, \cdot) = 0$, and with the boundary conditions (22)₂ one obtains

$$\sum_{j=1}^{j_0} [W(\mathcal{T}_{max}^j, \cdot, \cdot) - W(\mathcal{T}_{min}^j, \cdot, \cdot)]_{r_{j-1}}^{r_j} = 0. \quad (40)$$

Thus (36) yields

$$\partial_t \int_0^R (\mathcal{T}_{max} - \mathcal{T}_{min}) dr = \sum_{j=1}^{j_0} \int_{r_{j-1}}^{r_j} \partial_t (\mathcal{T}_{max}^j - \mathcal{T}_{min}^j) dr \leq 0, \quad (41)$$

i.e. $\int_0^R (\mathcal{T}_{max}(r, t) - \mathcal{T}_{min}(r, t)) dr$ is a monotonically decreasing function on any interval (t_{i-1}, t_i) . Using once more the continuity of \mathcal{T}_m this result holds on $[0, t^*)$, and since $t^* > 0$ is arbitrary, on $[0, \infty)$:

$$\int_0^R \left(\max_{|\mathbf{r}|=r} T(\mathbf{r}, t) - \min_{|\mathbf{r}|=r} T(\mathbf{r}, t) \right) r dr = \int_0^R (\mathcal{T}_{max}(r, t) - \mathcal{T}_{min}(r, t)) dr \searrow \text{ on } [0, \infty). \quad (42)$$

Remarks: 1. Concerning standard norms there holds clearly

$$\|T\|_{L^1(B_R)} = \int_{B_R} |T| d\mathbf{r} \leq 4\pi R \int_0^R \left(\max_{|\mathbf{r}|=r} T - \min_{|\mathbf{r}|=r} T \right) r dr, \quad (43)$$

which implies with (42) the bound

$$\|T(\cdot, t)\|_{L^1(B_R)} \leq 4\pi R \int_0^R \left(\max_{|\mathbf{r}|=r} T(\mathbf{r}, 0) - \min_{|\mathbf{r}|=r} T(\mathbf{r}, 0) \right) r \, dr \quad (44)$$

for all $t \geq 0$.

2. If the conductor is not a ball but a spherical shell with radii R_0 and R_1 one obtains instead of (42) monotonous decay of the quantity $\int_{R_0}^{R_1} (\mathcal{T}_{max}(r, t) - \mathcal{T}_{min}(r, t)) \, dr$. Note, moreover, that this result persists even in the case of moving (spherical) boundaries.

4 The steady case

The steady case is not yet ruled out by the (non-strict) decay result in the last section. The following argument, however, rules out this possibility. We assume in this section the flow field \mathbf{v} , the conductivity distribution λ , and a possible solution \mathcal{T} to be steady quantities. Hence the functions w , W , μ , and M from (18)–(20) do not vary with time. Starting point of the argument are the steady versions of eqs. (34) and (35). Their difference reads in the interval (r_{j-1}, r_j) :

$$\partial_r W(\mathcal{T}_{max}^j, \cdot) - \partial_r W(\mathcal{T}_{min}^j, \cdot) \leq \partial_r M(\partial_r \mathcal{T}_{max}^j, \cdot) - \partial_r M(\partial_r \mathcal{T}_{min}^j, \cdot). \quad (45)$$

We are going to show that the inequality (45) as well as the steady versions of (37) and (38) are in fact equalities. Integrating (45) over $[0, R]$ we obtain with (37)–(40)

$$0 = \sum_{j=1}^{j_0} [W(\mathcal{T}_{max}^j, \cdot) - W(\mathcal{T}_{min}^j, \cdot)]_{r_{j-1}}^{r_j} \leq \sum_{j=1}^{j_0} [M(\partial_r \mathcal{T}_{max}^j, \cdot) - M(\partial_r \mathcal{T}_{min}^j, \cdot)]_{r_{j-1}}^{r_j} \leq 0, \quad (46)$$

which would be false if any of the inequalities were strict. Therefore, integrating (45) over $[0, r]$ yields the equation

$$W(\mathcal{T}_{max}(r), r) - W(\mathcal{T}_{min}(r), r) = M(\partial_r \mathcal{T}_{max}(r), r) - M(\partial_r \mathcal{T}_{min}(r), r). \quad (47)$$

Writing (47) in the form

$$\int_{\mathcal{T}_{min}(r)}^{\mathcal{T}_{max}(r)} w(\tau, r) \, d\tau = \int_{\partial_r \mathcal{T}_{min}(r)}^{\partial_r \mathcal{T}_{max}(r)} \mu(\tau, r) \, d\tau \quad (48)$$

and using the bounds $|v_r| \leq C < \infty$, $\lambda \geq \lambda_0 > 0$ one obtains with (18) from (48) the differential inequality

$$\partial_r (\mathcal{T}_{max} - \mathcal{T}_{min}) \leq \frac{C}{\lambda_0} (\mathcal{T}_{max} - \mathcal{T}_{min}), \quad (49)$$

which, with initial-value $\mathcal{T}_{max}(0) - \mathcal{T}_{min}(0) = 0$, has the only solution $\mathcal{T}_{max} - \mathcal{T}_{min} \equiv 0$ and hence $\mathcal{T} \equiv 0$. So, nontrivial steady solutions of problem (22) do not exist.

5 Conclusions

It is shown in this paper that there are no steady or growing (in a mixed L^1 - L^∞ -norm), purely toroidal solutions of the kinematic dynamo problem, which satisfy the representation (18) and are regular in the sense of (26). Flow field and conductivity distribution are such that no poloidal magnetic field arises but are otherwise arbitrary. In particular, non-radial conductivity variations are allowed, but as it seems, they do not increase the propensity for dynamo action compared to a radially symmetric conductivity distribution. This result has been proved for a spherical conductor but is equally valid in plane geometry where the conductor is a plane layer and all functions are assumed to be horizontally periodic.

Note that our decay result refers to the toroidal scalar, not to the magnetic field itself. So, the possibility of a purely toroidal dynamo field with decaying scalar but increasing (non-radial) derivatives is not ruled out. On the other side, from a heuristic point of view, such solutions are not expected to be compatible with the parabolic character of the induction equation. Its rigorous exclusion, however, seems to be beyond the mathematical methods used in this article. Another question which is not answered by our decay result and which also seems not to be easily accessible by our means is the asymptotic value of the decaying quantity. A non-zero value is so far not excluded. But again, we do not expect such solutions since the corresponding non-zero stationary solutions have been excluded.

In deriving the decay result two nontrivial assumptions have been made, viz. the solution (18) of the constraint and the regularity condition (26) on T . Concerning the latter assumption one can expect that, using more sophisticated mathematical methods, condition (26) becomes redundant and that, in fact, stronger decay results, in particular decay to zero, can be proved. At least, in the case of a spherically symmetric conductivity distribution this has been demonstrated in (Kaiser 2009). Concerning the first assumption it is an open problem whether there are solutions of the constraint (17) which are not of type (18) and which allow magnetic field generation.

Finally, the reader might wonder if an invisible dynamo could not be obtained by a weaker assumption than that of a purely toroidal magnetic field. In fact, a dynamo solution with its poloidal component vanishing in the vacuum region but not in the conductor would be likewise invisible. In that case the poloidal field had to satisfy a double zero-condition at the boundary of the conductor. From a mathematical point of view this means solving an over-determined boundary value problem. So far this problem is not well understood. In a simpler context, viz. a spherically symmetric mean field model, invisible solutions of this type are known (Kaiser and Tilgner 2001), but not for the induction equation. The search for invisible solutions of the induction equation in a cylinder driven by a variable helical flow was (so far) not successful (Šimkanin and Tilgner 2008).

Appendix

In spherical geometry it is often useful to represent the gradient ∇ in \mathbb{R}^3 by one radial and two different non-radial derivatives, viz.

$$\partial_r := (\mathbf{r}/r) \cdot \nabla, \quad \nabla_{nr} := \nabla - (\mathbf{r}/r) \partial_r, \quad \Lambda := \mathbf{r} \times \nabla, \quad (\text{A1})$$

where $r = |\mathbf{r}|$. To facilitate calculations with these operators we collect in this appendix some relations concerning interchanging, dotting, and crossing with one another as well as with the radius vector.

Denoting the commutator of two quantities a and b by $[a, b] := ab - ba$ there holds

$$[\partial_r, r\nabla_{nr}] = [\partial_r, \Lambda] = 0 \quad (\text{A2})$$

and

$$[\partial_r, \mathbf{r}/r] = [\nabla_{nr}, r] = [\Lambda, r] = 0, \quad (\text{A3})$$

as can be verified directly from the definitions (A1).² We have, furthermore,

$$\nabla_{nr} \cdot \mathbf{r} = 2, \quad \nabla_{nr} \times \mathbf{r} = 0, \quad \Lambda \cdot \mathbf{r} = 0, \quad \Lambda \times \mathbf{r} = -2\mathbf{r}, \quad (\text{A4})$$

and for arbitrary functions f and g :

$$r^2 \nabla_{nr} f \cdot \nabla_{nr} g = \Lambda f \cdot \Lambda g, \quad r^2 \nabla_{nr} f \times \nabla_{nr} g = \Lambda f \times \Lambda g. \quad (\text{A5})$$

Concerning second derivatives there holds

$$\nabla_{nr} \cdot \Lambda = \Lambda \cdot \nabla_{nr} = 0, \quad (\text{A6})$$

$$r^2 \nabla_{nr} \times \nabla_{nr} = \Lambda, \quad \Lambda \times \Lambda = -\Lambda, \quad (\text{A7})$$

$$\nabla_{nr} \times \Lambda - (\mathbf{r}/r^2)\mathcal{L} = -\nabla_{nr}, \quad \Lambda \times \nabla_{nr} + (\mathbf{r}/r^2)\mathcal{L} = -\nabla_{nr}. \quad (\text{A8})$$

Remark: In plane geometry derivative operators can be defined analogously to (A1) with the Cartesian unit vector \mathbf{e}_z taking the role of \mathbf{r} . Relations (A2)–(A6) with the exception of (A4)_{1,4} then hold in an analogous way, but (A7), (A8), and (A4)_{1,4} do not. These nontrivial relations have no nontrivial counterparts in plane geometry.

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²We take the opportunity to correct a mistake in the appendix of (Kaiser 1995). The first commutator in (A.2) in this reference, $[r\nabla_{nr}, \Lambda] = 0$, does not hold in this generality. The conclusions in (Kaiser 1995) are, however, not affected by this false relation, since this relation has in fact never been used in calculations in this reference.

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