A Toroidal Magnetic Field Theorem

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Abstract

In the framework of magnetohydrodynamics the kinematic dynamo problem in a spherical fluid volume as well as in a plane layer is considered. On the premises of a purely toroidal magnetic field a nonlinear evolution equation for the toroidal scalar is derived. In this equation the flow field is constrained in such a way that no poloidal magnetic field can arise, but is otherwise arbitrary; the magnetic diffusivity is assumed to be spherically (horizontally, resp.) symmetric. Solutions of this problem are of particular interest since the magnetic field is confined to the fluid volume and therefore invisible to an external observer.

It is proved in this paper that the maximum norm of smooth solutions of this equation decays exponentially fast to zero. Thus, dynamo solutions, i.e. nondecaying solutions, of this type do not exist.

Key Words: Magnetohydrodynamics, dynamo theory, antidynamo theorem.

1 Introduction

It is common belief that the magnetic field of most stars and planets is generated by motions in an approximately spherically symmetric liquid conducting zone deep in the interior of the celestial body. The only trace of this process detectable for an external observer is the poloidal magnetic field component which extends into the surrounding vacuum, whereas the toroidal field component remains trapped in the conducting zone. So, a dynamo in a celestial body generating a purely toroidal magnetic field might remain invisible for an external observer.

However, the mechanism of field generation as described by the induction equation is usually explained in terms of an interaction of toroidal *and* poloidal magnetic field components (cf. [1]). So, several researchers in the field conjectured that dynamos generating a purely toroidal magnetic field do not exist [2, 3, 4, 5, 6]. This conjecture is similar to other well-established "antidynamo" theorems [7], e.g. Cowling's theorem excluding dynamos generating a purely axisymmetric magnetic field [8] or the "toroidal velocity theorem", which excludes field generation if the fluid motion is purely toroidal [9, 2]. The present situation, however, is more complicated than those described by these theorems. On the assumption of a purely

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toroidal magnetic field one obtains from the induction equation an evolution equation for the toroidal scalar T and, additionally, a constraint equation which is not automatically preserved by the evolution equation. In an earlier publication [10] a solution of this constraint has been proposed resulting in a nonlinear evolution equation for T.

The character of this equation is that of a second order parabolic equation with vanishing zeroth-order term and first-order terms which have neither pure advection form nor pure divergence form. It is this mixed form of the first-order terms, which prevents the straightforward application of maximum principles or L^2- (i.e. energy) or L^1- estimates, which are the mathematical essence of the above mentioned antidynamo results (cf. [11, 12, 13]). Instead, in [10] the monotonous decay of T in a mixed L^1-L^∞-norm, corresponding to the structure of the first-order terms, has been shown. From a mathematical point of view, however, the proof of this decay result cannot be considered as rigorous, as it depends on assumptions about T, which are not guaranteed for arbitrary (smooth) solutions of the governing equation. It is the aim of the present paper to make the arguments in [10] rigorous. Moreover, sharper decay results are derived, in particular, exponential decay of T to zero in the maximum norm.

The plan of the paper is as follows: In section 2 the mathematical framework of the dynamo problem is introduced, in particular, the induction equation and the poloidal/toroidal decomposition of solenoidal vector fields. Using this decomposition and a solution of the constraint the governing equation for the toroidal scalar T is derived, and previous results about solutions of this equation are reviewed. Section 3 presents a method to eliminate from a parabolic equation of mixed advection/divergence form those variables, which correspond to the advection part: Taking maxima and minima of a smooth solution with respect to these variables defines a "reduced function", which itself is a weak subsolution of a "reduced equation" of pure divergence form. In section 4 it is shown that decay results in [14] apply with minor modifications to solutions of this reduced equation. Finally, in section 5, the foregoing results are applied to the governing equation of the toroidal scalar, and corresponding decay results are formulated in the case of a spherical fluid domain as well as for a layer-like domain.

2 Mathematical setting and previous results

Given a volume $V \subset \mathbb{R}^3$ filled with a fluid of conductivity $\lambda > 0$, which is in prescribed motion according to a flow field $\mathbf{v}$, the kinematic dynamo problem asks for solutions $\mathbf{B}$, which do not decay in time, (so-called dynamo solutions) of the following initial-value problem:

$$(1)\qquad \begin{cases} \partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\lambda \nabla \times \mathbf{B})\,,\ \nabla \cdot \mathbf{B} = 0 & \text{in } V \times (0,\infty), \\ \nabla \times \mathbf{B} = 0\,, \quad \nabla \cdot \mathbf{B} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{V} \times (0,\infty), \\ \mathbf{B} \text{ continuous} & \text{in } \mathbb{R}^3 \times (0,\infty), \\ |\mathbf{B}(\mathbf{r},\cdot)| \to 0 & \text{for } |\mathbf{r}| \to \infty, \\ \mathbf{B}(\cdot\,,0) = \mathbf{B}_0\,, \quad \nabla \cdot \mathbf{B}_0 = 0 & \text{on } V \times \{t = 0\}. \end{cases}$$

The induction equation $(1)_1$ describes the generation of the magnetic field $\mathbf{B}$ by the motion of a conducting fluid. Outside the fluid volume we assume no further sources of magnetic field. Thus, $\mathbf{B}$ continues outside the fluid volume as a vacuum field, which vanishes at (spatial) infinity (cf. [12]).

The fluid volume V is in the following either a ball B_R of radius $R > 0$ or a plane layer $L := \mathbb{R}^2 \times [0, l]$ of thickness l. In the latter case $\mathbf{B}$ and $\mathbf{v}$ are assumed to be periodic in the "horizontal" variables x and y with periodicity cell $\mathcal{P} = [0, l_x] \times [0, l_y]$, and condition $(1)_4$ refers now to the "vertical" variable z, i.e. $|\mathbf{r}| \to \infty$ is replaced by $|z| \to \infty$. The conductivity is assumed to be spherically symmetric, i.e. $\lambda = \lambda(r,t)$, respectively $\lambda = \lambda(z,t)$ in the plane case, and bounded from below

$$\lambda \geq \lambda_0 > 0\,. \tag{2}$$

The flow field has a vanishing normal component at the boundary of the fluid volume, $\mathbf{n} \cdot \mathbf{v}|_{\partial V} = 0$, and satisfies some constraint (see below), but need not be divergence-free or symmetric in any sense. The plane and the spherical version of the dynamo problem have much in common, for instance, all known antidynamo theorems are equally valid in a ball or a plane layer (but not in a cylinder). The spherical case is closer to applications, whereas the plane case often serves as a toy model avoiding the complications due to spherical geometry. We deal with the plane case first.

The poloidal/toroidal decomposition reads in this case [15]:

$$\begin{aligned}\mathbf{B} &= \nabla \times (\nabla \times S\,\mathbf{e}_z) + \nabla \times T\,\mathbf{e}_z + \mathbf{b}(z)\\ &= (\partial_x\partial_z S + \partial_y T + b_x,\ \partial_y\partial_z S - \partial_x T + b_y,\ -\Delta_h S + b_z)^T.\end{aligned}$$

$\mathbf{e}_z$ is here the unit vector in z-direction, Δ_h means $\partial_x^2 + \partial_y^2$, and $\mathbf{b}$ is the horizontal mean of $\mathbf{B}$, $\mathbf{b} = \langle \mathbf{B} \rangle := \frac{1}{|\mathcal{P}|}\int_{\mathcal{P}} \mathbf{B}\,\mathrm{d}x\mathrm{d}y$, depending only on z (b_z being constant). Under the conditions $\langle S \rangle = \langle T \rangle = 0$ this decomposition is unique and allows the equivalent formulation of $(1)_1$ in terms of the toroidal scalar T, the poloidal one S, and the horizontal mean $\mathbf{b}$. If S and $\mathbf{b}$ are assumed to be zero, the horizontal mean of $(1)_1$, and the z-components of $(1)_1$ and of its curl yield:

$$\partial_z \langle v_z\, \partial_x T \rangle = \partial_z \langle v_z\, \partial_y T \rangle = 0\,, \tag{3}$$

$$\nabla T \times \nabla v_z \cdot \mathbf{e}_z = 0\,, \tag{4}$$

$$\Delta_h \big(\partial_t + \mathbf{v}\cdot\nabla - \nabla\cdot\lambda\nabla\big) T = \nabla \times (\nabla T \times \nabla v_z)\cdot \mathbf{e}_z\,. \tag{5}$$

Equations (3) and (4) constrain (the z-component of) the flow field in such a way that no poloidal magnetic field nor a mean field can arise. Unfortunately, these constraints are not preserved by the evolution equation (5), and must be incorporated in some way into (5).[1] A function $w(\tau, z, t)$ allowing the representation

$$v_z(x,y,z,t) = w(T(x,y,z,t), z, t) =: w_T(x,y,z,t) \tag{6}$$

for v_z clearly solves (3) and (4), and is motivated by the fact that (4) implies the existence of such a function locally if $\mathbf{e}_z \times \nabla T \neq 0$. With (6) the right-hand side in (5) can be reformulated:

$$\begin{gathered}\mathbf{e}_z \times \nabla \cdot [\nabla T \times \nabla w_T] = -\mathbf{e}_z \times \nabla \cdot [\partial_z w|_{\tau=T}\ \mathbf{e}_z \times \nabla T]\\ = -(\mathbf{e}_z \times \nabla)\cdot(\mathbf{e}_z \times \nabla)\Big[\int_0^T \partial_z w\,\mathrm{d}\tau\Big] = -\Delta_h[\partial_z W|_{\tau=T}] = -\Delta_h[\partial_z W_T - v_z \partial_z T]\end{gathered}$$

[1] This is different to the divergence-constraint in $(1)_1$. Imposing the constraint on the initial value of the induction equation guarantees already a divergence-free solution.

with $W(T,z,t) := \int_0^T w(\tau,z,t)\,\mathrm{d}\tau$ and $W_T(x,y,z,t) := W(T(x,y,z,t),z,t)$. Thus (5) takes the form

$$\Delta_h\big[\big(\partial_t + \mathbf{v}\cdot\nabla_h - \nabla\cdot\lambda\nabla\big)T + \partial_z W_T\big] = 0\,, \tag{7}$$

where $\nabla_h = \nabla - e_z\partial_z$. Note that Δ_h can be removed from eq. (7) if the bracket has zero mean.

Since T vanishes outside the fluid layer the problem for purely toroidal dynamo fields reads now:

$$\begin{cases} \partial_t T = \partial_z(\lambda\,\partial_z T) + \lambda\nabla_h\cdot\nabla_h T - \partial_z W_T - \mathbf{v}\cdot\nabla_h T & \\ \qquad +\langle\partial_z W_T + \mathbf{v}\cdot\nabla_h T\rangle & \text{in } L\times(0,\infty), \\ T = 0 & \text{on } \mathrm{I\!R}^2\times\{z=0,l\}\times(0,\infty), \\ T(\cdot\,,0) = T_0\,,\quad \langle T_0\rangle = 0 & \text{on } L\times\{t=0\}. \end{cases} \tag{8}$$

In [10] the decay of smooth solutions of problem (8) is based on the following reasoning: Consider the horizontal maximum of T,

$$T_{max}(z,t) := \max_{x,y} T(x,y,z,t)$$

and the associated "path" $\mathbf{r}_h^{max}(z,t) = (x^{max}(z,t), y^{max}(z,t))$, where the maximum is attained:

$$T_{max}(z,t) = T(\mathbf{r}_h^{max}(z,t),z,t).$$

At maximum points we have clearly

$$\nabla_h T(\mathbf{r}_h^{max},z,t) = 0\,,\quad \Delta_h T(\mathbf{r}_h^{max},z,t) \le 0\,.$$

Moreover, we have

$$\partial_z T_{max}(z,t) = \partial_z T(\mathbf{r}_h^{max},z,t) + \nabla_h T(\mathbf{r}_h^{max},z,t)\,\partial_z\mathbf{r}_h^{max}(z,t) = \partial_z T(\mathbf{r}_h^{max},z,t)\,,$$

and similarly,

$$\partial_t T_{max} = \partial_t T|_{\mathbf{r}_h=\mathbf{r}_h^{max}}\,,\qquad \partial_z^2\,T_{max} \ge \partial_z^2\,T|_{\mathbf{r}_h=\mathbf{r}_h^{max}}\,.$$

Thus, evaluating $(8)_1$ at $\mathbf{r}_h^{max}$ yields the inequality for T_{max}:

$$\partial_t T_{max} \le \partial_z(\lambda\,\partial_z T_{max}) - \partial_z W_{T_{max}} + m \tag{9}$$

with $m = m(z,t)$ denoting the mean value in $(8)_1$. Analogous arguments for $T_{min} := \min_{x,y} T$ yield the inequality

$$\partial_t T_{min} \ge \partial_z(\lambda\,\partial_z T_{min}) - \partial_z W_{T_{min}} + m\,. \tag{10}$$

Note that $T_{max} - T_{min}$ is nonnegative due to the zero-mean condition. Thus, integrating (9) and (10) with respect to z over the interval $[0,l]$ and using the boundary conditions for T one obtains the decay result:

$$\partial_t \int_0^l (T_{max} - T_{min})\,\mathrm{d}z \le 0\,. \tag{11}$$

The derivation of (11) relies on the existence of smooth functions $\mathbf{r}_h^{max/min}(z,t)$. These functions, however are not well-defined. In general, they are not unique and cannot be defined as continuous functions (cf. fig.1 in [10]). In [10] inequality (11) has been proved on the assumption of piecewise smooth functions $\mathbf{r}_h^{max/min}$. But even this is not guaranteed for solutions of (8), not even for arbitrarily smooth solutions. It is the aim of the present paper to prove (in fact more than) (11) without falling back on functions like $\mathbf{r}_h^{max/min}$.

In the spherical case the poloidal/toroidal decomposition reads [12, 16]:

$$\mathbf{B} = \nabla \times (\nabla \times S\mathbf{r}) + \nabla \times T\mathbf{r} = -\nabla \times \Lambda S - \Lambda T \tag{12}$$

with $\Lambda := \mathbf{r}\times\nabla$. The mean $\langle\,\cdot\,\rangle$ is now taken over spheres S_r of radius r, $\langle\,\cdot\,\rangle := (4\pi r^2)^{-1}\int_{S_r}\cdot\,\mathrm{d}s$, a mean field does not arise, and $\mathcal{L} := \Lambda\cdot\Lambda$ denotes now the Laplace-Beltrami-operator on the unit sphere. On the assumption of a purely toroidal magnetic field the equations corresponding to (4), (5) read

$$\nabla T \times \nabla(\mathbf{v}\cdot\mathbf{r})\cdot\mathbf{r} = 0\,, \tag{13}$$

$$\mathcal{L}\big(\partial_t + \mathbf{v}\cdot\nabla - \lambda\,\Delta - \frac{\lambda'}{r}\,(1 + r\partial_r)\big)T = \Lambda\cdot(\nabla T \times \nabla(\mathbf{v}\cdot\mathbf{r})) \tag{14}$$

with $\partial_r := (\mathbf{r}/r)\cdot\nabla$ and $\lambda' := \partial_r\lambda$. Introducing the variable $\mathcal{T} := rT$, (13) takes the form

$$\nabla\mathcal{T} \times \nabla v_r \cdot \mathbf{r} = 0\,,$$

and is solved by a function $w(\tau, r, t)$ allowing the representation for v_r:

$$v_r(\mathbf{r},t) = w(\mathcal{T}(\mathbf{r},t), r, t) =: w_{\mathcal{T}}(\mathbf{r},t)\,. \tag{15}$$

With the identity

$$\nabla T \times \nabla(rv_r) = \frac{1}{r}\,\Lambda T \times \Lambda v_r - \frac{1}{r}\,\partial_r(rv_r)\,\Lambda T + \partial_r T\,\Lambda v_r$$

the right-hand side of eq. (14) can be rewritten using (15) as [2]

$$\begin{aligned}\Lambda\cdot\big[\nabla T \times \nabla(rv_r)\big] &= \Lambda\cdot\big[-\frac{1}{r}\,\partial_r(rv_r)\,\Lambda T - \frac{1}{r}\,(rv_r)\,\partial_r\Lambda T + \Lambda(v_r\,\partial_r T)\big]\\ &= -\frac{1}{r}\,\Lambda\cdot\partial_r\big[v_r\,\Lambda(rT)\big] + \Lambda\cdot\Lambda\,(v_r\,\partial_r T) = -\mathcal{L}\big[\partial_r W_{\mathcal{T}} - v_r\,\partial_r T\big]\end{aligned}$$

with $W(\mathcal{T}, r, t) := \int_0^{\mathcal{T}} w(\tau, r, t)\,\mathrm{d}\tau$ and $W_{\mathcal{T}}(\mathbf{r},t) := W(\mathcal{T}(\mathbf{r},t), r, t)$. Thus, writing (14) in terms of $\mathcal{T}$ and discarding $\mathcal{L}$, we arrive at a problem analogous to (8):

$$\begin{cases} \partial_t\mathcal{T} = \partial_r(\lambda\,\partial_r\mathcal{T}) + \lambda\nabla_{nr}\cdot\nabla_{nr}\mathcal{T} - \partial_r W_{\mathcal{T}} & \\ \qquad -\mathbf{v}\cdot\nabla_{nr}\mathcal{T} + \langle\partial_r W_{\mathcal{T}} + \mathbf{v}\cdot\nabla_{nr}\mathcal{T}\rangle & \text{in } B_R \times (0,\infty),\\ \mathcal{T} = 0 & \text{on } (S_R \cup \{r=0\}) \times (0,\infty),\\ \mathcal{T}(\cdot\,,0) = \mathcal{T}_0\,,\quad \langle\mathcal{T}_0\rangle = 0 & \text{on } B_R \times \{t=0\}\,. \end{cases} \tag{16}$$

[2]Note that the operator Λ interchanges with r as well as with ∂_r.

∇_{nr} denotes here the non-radial gradient $\nabla - (\mathbf{r}/r)\partial_r$.

Questions of existence and uniqueness of solutions have been treated in [17] for the spherical case (16).[3] According to this reference the following regularity assumptions:[4]

$$\mathbf{v}_{nr} := \mathbf{v} - v_r(\mathbf{r}/r) \in C^{\alpha,\alpha/2}(\overline{B_R} \times [0,t_0])\,, \quad \lambda \in C^{1+\alpha,(1+\alpha)/2}(\overline{B_R} \times [0,t_0])\,,$$
$$\mathcal{T}_0 \in C^{2+\alpha}(\overline{B_R})$$

for any $t_0 > 0$,

$$w(0,r,t) = 0\,, \quad \frac{\partial^{k+l+m}}{\partial\tau^k \partial r^l \partial t^m}\, w \text{ exists and is continuous on } \mathbb{R} \times [0,R] \times [0,\infty)$$

for any $k+l+m \leq 3$, $m \leq 1$, $l \leq 2-m$, together with the compatibility conditions

$$\mathcal{T}_0 = \partial_r(\lambda \partial_r \mathcal{T}_0) = 0 \quad \text{ on } S_R$$

and the ellipticity condition (2) guarantee a unique local solution

$$\mathcal{T} \in C^{2+\alpha,(2+\alpha)/2}(\overline{B_R} \times [0,t^*)) \tag{17}$$

of the *nonlinear* problem (16). Moreover, if the maximal time of existence $t^* < \infty$ the solution blows up in the $1+\alpha$ - norm:

$$\lim_{t \to t^*} \|\mathcal{T}\|_{1+\alpha,(1+\alpha)/2} < \infty\,.$$

Analogous results hold also in the (simpler) plane case and can be proved along the lines of ref. [17].

On the other hand the toroidal magnetic field $\mathbf{B}_T$ belonging to a reasonable solution T of (8) ($\mathcal{T}$ of (16), resp.) should also be a solution of the original *linear* system (1) with additional property $\mathbf{B}_T \cdot \nabla v_z = 0$ ($\mathbf{B}_T \cdot \nabla(\mathbf{v} \cdot \mathbf{r}) = 0$, resp.). Defining $\mathbf{B}_T =: (T_y, -T_x, 0)$ this system takes in the plane case the form

$$\partial_t T_x = \nabla \cdot (\lambda \nabla T_x) - \partial_x(v_x\, T_x) - \partial_z(v_z\, T_x) - \partial_x(v_y\, T_y)\,,$$
$$\partial_t T_y = \nabla \cdot (\lambda \nabla T_y) - \partial_y(v_y\, T_y) - \partial_z(v_z\, T_y) - \partial_y(v_x\, T_x)\,,$$

which is of type (1.1) in [18, chap. VII]. In the spherical case an analogous system is obtained for the nonradial derivatives of $\mathcal{T}$. According to Theorem 3.1 in [18, chap. VII] condition (2), mild regularity assumptions on the coefficients λ and $\mathbf{v}$ (e.g. $\lambda, \mathbf{v} \in C^{\alpha,\alpha/2}(\overline{V} \times [0,t_0])$ is enough), and appropriate conditions on the initial value already imply

$$\mathbf{B}_T \in C^{\alpha,\alpha/2}(\overline{V} \times [0,t_0]) \tag{18}$$

for any $t_0 > 0$. Improved regularity of the coefficients and initial value leads to improved regularity of $\mathbf{B}_T$ (cf. Theorem 4.1 in [18, chap. VII]). Property (18) implies already boundedness of the horizontal derivatives of T (the nonradial derivatives of $\mathcal{T}$, resp.). Boundedness

[3] In fact, the problem considered in [17] differs slightly from (16), since the author of [17] uses T instead of $\mathcal{T} = rT$ (which is more appropriate in our case) as dynamic variable.

[4] Concerning (uniform) Hölder continuity and Hölder norms in parabolic problems we refer to [17] or [18].

of the vertical (radial, resp.) derivative is implied by $\mathbf{B}_T \in C^{1+\alpha,(1+\alpha)/2}(\overline{V} \times [0,t_0])$. In fact, observing that $\Delta_h^{-1}(\mathbf{e}_z \times \nabla)\cdot$ is a bounded operator $V^2 \to V$ with

$$V := \{f : \mathbb{R}^2 \to \mathbb{R} \,|\, f\, \mathcal{P}\text{−periodic}, \langle f\rangle = 0, \|f\|_{C^\alpha} < \infty\},$$

one obtains

$$\partial_z T = \Delta_h^{-1}(\mathbf{e}_z \times \nabla) \cdot \partial_z \mathbf{B}_T \in C^{\alpha,\alpha/2}(\overline{L} \times [0,t_0])$$

and an analogous result holds in the spherical case (cf. [19]). Thus, T resp. $\mathcal{T}$ remain bounded in $C^{1+\alpha,(1+\alpha)/2}(\overline{V} \times [0,t_0])$ for any $t_0 > 0$. In conclusion, any regular solution of problem (8) or (16), whose associated magnetic field $\mathbf{B}_T$ solves the original problem (1) with sufficiently regular data is global in time. Solutions of this type are considered henceforth.

3 A reduction method for parabolic equations

Let us consider a parabolic equation of the form

$$\partial_t T = \sum_{i,j=1}^{m} \partial_{x_i}(a_{ij}\, \partial_{x_j} T) + \sum_{k,l=1}^{n} b_{kl}\, \partial_{y_k}\partial_{y_l} T - \nabla_x \cdot \mathbf{W}_T - \mathbf{v} \cdot \nabla_y T + c\,, \tag{19}$$

which distinguishes two types of spatial variables: $\mathbf{x} \in G$ and $\mathbf{y} \in H$ with open bounded sets $G \subset \mathbb{R}^m$ and $H \subset \mathbb{R}^n$. Gradients with respect to $\mathbf{x}$ and $\mathbf{y}$ are denoted by ∇_x and ∇_y, respectively; ∇ refers to both variables. T is assumed to be a smooth solution of (19),

$$T \in C_1^2(\overline{G} \times \overline{H} \times [0,t_0])\,, \qquad t_0 > 0\,, \tag{20}$$

i.e. T, $\partial_t T$, ∇T, and $\nabla\nabla T$ are all continuous functions on $(\overline{G} \times \overline{H} \times [0,t_0])$. The various coefficients in eq. (19) depend on different variables: $a_{ij} = a_{ij}(\mathbf{x},t)$, $b_{kl} = b_{kl}(T,\mathbf{x},\mathbf{y},t)$, $\mathbf{W} = \mathbf{W}(\tau,\mathbf{x},t)$, $\mathbf{v} = \mathbf{v}(T,\mathbf{x},\mathbf{y},t)$, and $c = c(\mathbf{x},t)$. $\mathbf{W}_T$ means again $\mathbf{W}(T(\cdot),\cdot)$, thus $\nabla_x \cdot \mathbf{W}_T = \nabla_x \cdot \mathbf{W}|_{\tau=T} + \partial_\tau \mathbf{W}|_{\tau=T} \cdot \nabla T$ being of divergence form. (a_{ij}) and (b_{kl}) are symmetric, positive definite matrices uniformly bounded away from zero. All coefficients are assumed to be sufficiently smooth to insure a solution of type (20); however, in this section we will make use only of boundedness of a_{ij}, $\mathbf{W}$, and $\partial_\tau \mathbf{W} =: \mathbf{w}$ on their respective domains.

In order to eliminate the advection term from eq. (19) we define

$$T_{max}(\mathbf{x},t) := \max_{\mathbf{y}\in\overline{H}} T(\mathbf{x},\mathbf{y},t)\,, \qquad T_{min}(\mathbf{x},t) := \min_{\mathbf{y}\in\overline{H}} T(\mathbf{x},\mathbf{y},t)\,, \tag{21}$$

and for fixed $(\mathbf{x},t) \in \overline{G} \times [0,t_0]$ the set of “extremal points”

$$\{\mathbf{y}_m(\mathbf{x},t)\} := \{\mathbf{y} \in \overline{H} : T(\mathbf{x},\mathbf{y},t) = T_m(\mathbf{x},t)\}$$

with $m = \max$ or $\min$. Thus, for any $\mathbf{y}_m(\mathbf{x},t) \in \{\mathbf{y}_m(\mathbf{x},t)\}$ we have

$$T_m(\mathbf{x},t) = T(\mathbf{x},\mathbf{y}_m(\mathbf{x},t),t) \qquad m \in \{\max,\min\}\,. \tag{22}$$

Despite its pointwise definition T_m has some smoothness:

Lemma 1 *Let T, $\nabla_x T$, and $\partial_t T$ be continuous on $\overline{G} \times \overline{H} \times [0, t_0]$, then $T_m : \overline{G} \times [0, t_0] \to \mathbb{R}$ with $m \in \{\max, \min\}$ defined by (21) is Lipschitz continuous, a.e. differentiable, and there holds a.e.*

$$(23) \qquad \begin{cases} \nabla_x T_m(\mathbf{x}, t) = \nabla_x T(\mathbf{x}, \mathbf{y}, t)|_{\mathbf{y}=\mathbf{y}_m(\mathbf{x},t)} , \\ \partial_t T_m(\mathbf{x}, t) = \partial_t T(\mathbf{x}, \mathbf{y}, t)|_{\mathbf{y}=\mathbf{y}_m(\mathbf{x},t)} \end{cases}$$

with arbitrary extremal points $\mathbf{y}_m(\mathbf{x}, t) \in \{\mathbf{y}_m(\mathbf{x}, t)\}$.

PROOF: Let $L > 0$ such that

$$|\nabla_x T| < L , \quad |\partial_t T| < L \quad \text{ on } \overline{G} \times \overline{H} \times [0, t_0]$$

and w.l.o.g. $T_m(\mathbf{x}, t) \geq T_m(\tilde{\mathbf{x}}, \tilde{t})$. With $\mathbf{y}_m(\mathbf{x}, t) \in \{\mathbf{y}_m(\mathbf{x}, t)\}$, (22) and (21) follows:

$$\begin{aligned} |T_{max}(\mathbf{x}, t) - T_{max}(\tilde{\mathbf{x}}, \tilde{t})| &= T_{max}(\mathbf{x}, t) - T_{max}(\tilde{\mathbf{x}}, \tilde{t}) \\ &= T(\mathbf{x}, \mathbf{y}_{max}(\mathbf{x}, t), t) - T(\tilde{\mathbf{x}}, \mathbf{y}_{max}(\tilde{\mathbf{x}}, \tilde{t}), \tilde{t}) \\ &\leq T(\mathbf{x}, \mathbf{y}_{max}(\mathbf{x}, t), t) - T(\tilde{\mathbf{x}}, \mathbf{y}_{max}(\mathbf{x}, t), \tilde{t}) \leq L\, |(\mathbf{x} - \tilde{\mathbf{x}},\, t - \tilde{t})| \end{aligned}$$

and

$$\begin{aligned} |T_{min}(\mathbf{x}, t) - T_{min}(\tilde{\mathbf{x}}, \tilde{t})| &= T(\mathbf{x}, \mathbf{y}_{min}(\mathbf{x}, t), t) - T(\tilde{\mathbf{x}}, \mathbf{y}_{min}(\tilde{\mathbf{x}}, \tilde{t}), \tilde{t}) \\ &\leq T(\mathbf{x}, \mathbf{y}_{min}(\tilde{\mathbf{x}}, \tilde{t}), t) - T(\tilde{\mathbf{x}}, \mathbf{y}_{min}(\tilde{\mathbf{x}}, \tilde{t}), \tilde{t}) \leq L\, |(\mathbf{x} - \tilde{\mathbf{x}},\, t - \tilde{t})| \, . \end{aligned}$$

Thus, T_m is Lipschitz continuous and according to Rademacher's theorem a.e. differentiable.

Consider next forward and backward difference quotients with respect to t. We have with $h > 0$:

$$\begin{aligned} \frac{1}{h}\big(T_{max}(\mathbf{x}, t+h) - T_{max}(\mathbf{x}, t)\big) &= \frac{1}{h}\big(T(\mathbf{x}, \mathbf{y}_{max}(\mathbf{x}, t+h), t+h) \\ -T(\mathbf{x}, \mathbf{y}_{max}(\mathbf{x}, t), t)\big) &\geq \frac{1}{h}\big(T(\mathbf{x}, \mathbf{y}_{max}(\mathbf{x}, t), t+h) - T(\mathbf{x}, \mathbf{y}_{max}(\mathbf{x}, t), t)\big) \, . \end{aligned}$$

As the last expression has a well-defined limit for $h \to 0$ we obtain

$$(24) \qquad \liminf_{h \searrow 0} \frac{1}{h}\big(T_{max}(\mathbf{x}, t+h) - T_{max}(\mathbf{x}, t)\big) \geq \partial_t T(\mathbf{x}, \mathbf{y}, t)|_{\mathbf{y}=\mathbf{y}_{max}(\mathbf{x},t)} \, .$$

On the other hand,

$$\frac{1}{h}\big(T_{max}(\mathbf{x}, t) - T_{max}(\mathbf{x}, t-h)\big) \leq \frac{1}{h}\big(T(\mathbf{x}, \mathbf{y}_{max}(\mathbf{x}, t), t) - T(\mathbf{x}, \mathbf{y}_{max}(\mathbf{x}, t), t-h)\big) \, ,$$

which implies

$$(25) \qquad \limsup_{h \searrow 0} \frac{1}{h}\big(T_{max}(\mathbf{x}, t) - T_{max}(\mathbf{x}, t-h)\big) \leq \partial_t T(\mathbf{x}, \mathbf{y}, t)|_{\mathbf{y}=\mathbf{y}_{max}(\mathbf{x},t)} \, .$$

So, if T_{max} is differentiable in $(\mathbf{x}, t)$, (24) and (25) imply

$$\partial_t T_{max}(\mathbf{x}, t) = \partial_t T(\mathbf{x}, \mathbf{y}, t)|_{\mathbf{y}=\mathbf{y}_{max}(\mathbf{x},t)} \, .$$

For T_{min} one obtains similarly with $h > 0$

$$\limsup_{h\searrow 0} \frac{1}{h} \big(T_{min}(\mathbf{x}, t+h) - T_{min}(\mathbf{x}, t)\big) \le \partial_t T(\mathbf{x}, \mathbf{y}, t)|_{\mathbf{y}=\mathbf{y}_{min}(\mathbf{x},t)}\,,$$
$$\liminf_{h\searrow 0} \frac{1}{h} \big(T_{min}(\mathbf{x}, t) - T_{min}(\mathbf{x}, t-h)\big) \ge \partial_t T(\mathbf{x}, \mathbf{y}, t)|_{\mathbf{y}=\mathbf{y}_{min}(\mathbf{x},t)}\,,$$

which imply in the case of differentiability again

$$\partial_t T_{min}(\mathbf{x}, t) = \partial_t T(\mathbf{x}, \mathbf{y}, t)|_{\mathbf{y}=\mathbf{y}_{min}(\mathbf{x},t)}\,.$$

Analogous results hold for the spatial derivatives ∂_{x_i}, $i = 1, \ldots, m$.

□

If T is of class (20) we have at extremal points $\mathbf{y}_m$ clearly $\nabla_y T|_{\mathbf{y}=\mathbf{y}_m} = 0$ and

$$\sum_{k,l=1}^{n} (b_{kl}\, \partial_{y_k} \partial_{y_l} T)|_{\mathbf{y}=\mathbf{y}_{max}} \le 0\,, \quad \sum_{k,l=1}^{n} (b_{kl}\, \partial_{y_k} \partial_{y_l} T)|_{\mathbf{y}=\mathbf{y}_{min}} \ge 0\,.$$

Therefore, evaluating eq. (19) at extremal points yields the inequalities

$$\text{(26)} \qquad \begin{cases} \partial_t T_{max} \le \displaystyle\sum_{i,j=1}^{m} \big(\partial_{x_i}(a_{ij}\, \partial_{x_j} T)\big)|_{\mathbf{y}=\mathbf{y}_{max}} - \nabla_x \cdot \mathbf{W}_{T_{max}} + c\,, \\ \partial_t T_{min} \ge \displaystyle\sum_{i,j=1}^{m} \big(\partial_{x_i}(a_{ij}\, \partial_{x_j} T)\big)|_{\mathbf{y}=\mathbf{y}_{min}} - \nabla_x \cdot \mathbf{W}_{T_{min}} + c\,. \end{cases}$$

We used here Lemma 1 on the left-hand side and for rewriting the second term on the right-hand side:

$$\nabla_x \cdot \mathbf{W}_T|_{\mathbf{y}=\mathbf{y}_m} = \nabla_x \cdot \mathbf{W}|_{\tau=T_m} + \partial_\tau \mathbf{W} \cdot \nabla_x T_m = \nabla_x \cdot \mathbf{W}_{T_m}\,.$$

The inequalities (26) cannot completely be expressed in terms of T_m, since second-order derivatives of T_m do not make sense. In fact, more than Lipschitz continuity and hence differentiability a.e. cannot be expected (cf. fig. 1 in [10]). In a weak form, however, (26) can be reduced to inequalities for T_m. For this purpose we define the set of nonnegative test functions by

$$C_{0,+}^{\infty} := C_{0,+}^{\infty}(G \times (-\infty, t_0)) := \big\{\phi \in C_0^{\infty}(G \times (-\infty, t_0)) \,|\, \phi \ge 0\big\}$$

and prove

Lemma 2 *Let T be of class* (20)*, T_m and $\mathbf{y}_m$ as in Lemma 1 with $m \in \{\max, \min\}$, and (a_{ij}) as explained after* (20)*. Then*

$$\sum_{i,j=1}^{m} \int_G \big(\partial_{x_i}(a_{ij}\, \partial_{x_j} T)\big)|_{\mathbf{y}=\mathbf{y}_{max}}\, \phi\, \mathrm{d}x \le - \sum_{i,j=1}^{m} \int_G a_{ij}\, \partial_{x_j} T_{max}\, \partial_{x_i} \phi\, \mathrm{d}x\,,$$
$$\sum_{i,j=1}^{m} \int_G \big(\partial_{x_i}(a_{ij}\, \partial_{x_j} T)\big)|_{\mathbf{y}=\mathbf{y}_{min}}\, \phi\, \mathrm{d}x \ge - \sum_{i,j=1}^{m} \int_G a_{ij}\, \partial_{x_j} T_{min}\, \partial_{x_i} \phi\, \mathrm{d}x$$

for a.e. $t \in (0, t_0)$ and arbitrary $\phi \in C_{0,+}^{\infty}(G \times (-\infty, t_0))$.

Proof: Only the case $m = max$ is proved, the other case follows with minor modifications. Denoting forward and backward difference quotients in direction $\mathbf{n}$ with step width $h > 0$ by

$$D^h_{\mathbf{x}\cdot\mathbf{n}} u(\mathbf{x}) := \frac{1}{h}\big(u(\mathbf{x}+h\mathbf{n}) - u(\mathbf{x})\big) \text{ and } D^{-h}_{\mathbf{x}\cdot\mathbf{n}} u(\mathbf{x}) := \frac{1}{h}\big(u(\mathbf{x}) - u(\mathbf{x}-h\mathbf{n})\big),$$

respectively, one obtains with $\mathbf{y}_m(\mathbf{x},t) \in \{\mathbf{y}_m(\mathbf{x},t)\}$, (22) and (21):

$$(27)\quad \begin{cases} \big(D^{-h}_{\mathbf{x}\cdot\mathbf{n}} D^h_{\mathbf{x}\cdot\mathbf{n}} T(\mathbf{x},\mathbf{y},t)\big)|_{\mathbf{y}=\mathbf{y}_{max}(\mathbf{x},t)} \\ = \frac{1}{h^2}\big(T(\mathbf{x}+h\mathbf{n},\mathbf{y}_{max}(\mathbf{x},t),t) + T(\mathbf{x}-h\mathbf{n},\mathbf{y}_{max}(\mathbf{x},t),t) - 2\,T(\mathbf{x},\mathbf{y}_{max}(\mathbf{x},t),t)\big) \\ \le \frac{1}{h^2}\big(T(\mathbf{x}+h\mathbf{n},\mathbf{y}_{max}(\mathbf{x}+h\mathbf{n},t),t) + T(\mathbf{x}-h\mathbf{n},\mathbf{y}_{max}(\mathbf{x}-h\mathbf{n},t),t) \\ \qquad -2\,T(\mathbf{x},\mathbf{y}_{max}(\mathbf{x},t),t)\big) \\ = D^{-h}_{\mathbf{x}\cdot\mathbf{n}} D^h_{\mathbf{x}\cdot\mathbf{n}} T_{max}(\mathbf{x},t)\,. \end{cases}$$

(27) implies the matrix

$$(28)\qquad \big(D^{-h}_{x_i} D^h_{x_j} T_{max} - (D^{-h}_{x_i} D^h_{x_j} T)|_{\mathbf{y}=\mathbf{y}_{max}}\big)$$

being positive semidefinite. Thus, contracting (28) with (a_{ij}) and multiplying with $\phi \in C^\infty_{0,+}$ yields

$$\begin{aligned} &\sum_{i,j=1}^m (D^{-h}_{x_i} D^h_{x_j} T)|_{\mathbf{y}=\mathbf{y}_{max}}\, a_{ij}\,\phi \le \sum_{i,j=1}^m D^{-h}_{x_i} D^h_{x_j} T_{max}\, a_{ij}\,\phi \\ &= -\sum_{i,j=1}^m D^h_{x_j} T_{\max}\, D^h_{x_i}(a_{ij}\phi) \\ &\quad -\sum_{i,j=1}^m \frac{1}{h}\left[(D^h_{x_j} T_{\max})|_{(\mathbf{x}-h\mathbf{e}_i,t)}\, a_{ij}\,\phi - D^h_{x_j} T_{\max}\,(a_{ij}\phi)|_{(\mathbf{x}+h\mathbf{e}_i,t)}\right], \end{aligned}$$

and integration over G with $h < \operatorname{dist}\big(\operatorname{supp}(\phi), \partial(G\times(-\infty,t_0))\big)$:

$$\sum_{i,j=1}^m \int_G (D^{-h}_{x_i} D^h_{x_j} T)|_{\mathbf{y}=\mathbf{y}_{max}}\, a_{ij}\,\phi\, \mathrm{d}x \le -\sum_{i,j=1}^m \int_G D^h_{x_j} T_{max}\, D^h_{x_i}(a_{ij}\phi)\,\mathrm{d}x\,.$$

Observe now that (20) implies for fixed $\mathbf{y}_{max}(\mathbf{x},t)$ the pointwise limit on $\overline{G}\times[0,t_0]$:

$$(D^{-h}_{x_i} D^h_{x_j} T)|_{\mathbf{y}=\mathbf{y}_{max}} \to (\partial_{x_i}\partial_{x_j} T)|_{\mathbf{y}=\mathbf{y}_{max}} \qquad \text{for } h \to 0\,,$$

and Lemma 1 the limit a.e. on $\overline{G}\times[0,t_0]$:

$$D^h_{x_j} T_{max} \to \partial_{x_j} T_{max} \qquad \text{for } h \to 0\,.$$

Moreover, there are constants L, $\tilde{L} > 0$ such that

$$\operatorname*{ess\,sup}_{\overline{G}\times[0,t_0]} |D^h_{x_j} T_{max}| \le L\,, \qquad \max_{\overline{G}\times\overline{H}\times[0,t_0]} |D^{-h}_{x_i} D^h_{x_j} T| \le \tilde{L}\,.$$

Thus, Lebesgue's theorem yields

$$\sum_{i,j=1}^m \int_G (\partial_{x_i}\partial_{x_j} T)|_{\mathbf{y}=\mathbf{y}_{max}}\, a_{ij}\,\phi\,\mathrm{d}x \le -\sum_{i,j=1}^m \int_G \partial_{x_j} T_{max}\, \partial_{x_i}(a_{ij}\phi)\,\mathrm{d}x\,.$$

Lemma 1 implies, moreover, the equation

$$\sum_{i,j=1}^{m} \int_G \partial_{x_i} a_{ij} (\partial_{x_j} T)|_{\mathbf{y}=\mathbf{y}_{max}} \, \phi \, \mathrm{d}x = \sum_{i,j=1}^{m} \int_G \partial_{x_j} T_{max} \, \partial_{x_i} a_{ij} \, \phi \, \mathrm{d}x \,,$$

which concludes the proof.

□

Lemma 2 enables us to write (26) in a weak form as inequalities for T_m:

$$(29) \quad \begin{cases} \int_0^{t_0} \int_G T_m \, \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t + \int_G T_m(\cdot, 0) \phi(\cdot, 0) \, \mathrm{d}x \\ \gtreqless \sum_{i,j=1}^{m} \int_0^{t_0} \int_G a_{ij} \, \partial_{x_j} T_m \, \partial_{x_i} \phi \, \mathrm{d}x \, \mathrm{d}t - \int_0^{t_0} \int_G \mathbf{W}_{T_m} \cdot \nabla \phi \, \mathrm{d}x \, \mathrm{d}t - \int_0^{t_0} \int_G c \, \phi \, \mathrm{d}x \, \mathrm{d}t \,, \end{cases}$$

with the upper inequality referring to $m = max$ and the lower one to $m = min$. Introducing

$$(30) \qquad \delta T := T_{max} - T_{min}$$

the inequalities (29) can be combined to a single one. Writing

$$\mathbf{W}(T_{max}, \cdot, \cdot) - \mathbf{W}(T_{min}, \cdot, \cdot) = \frac{\mathbf{W}(T_{max}, \cdot, \cdot) - \mathbf{W}(T_{min}, \cdot, \cdot)}{T_{max} - T_{min}} \, \delta T$$

and defining the bounded function

$$(31) \qquad \boldsymbol{\omega}(\mathbf{x}, t) := \frac{\mathbf{W}(\tau, \mathbf{x}, t) - \mathbf{W}(\sigma, \mathbf{x}, t)}{\tau - \sigma} \Big|_{\tau = T_{max}(\mathbf{x},t), \, \sigma = T_{min}(\mathbf{x},t)}$$

we obtain from (29)

$$(32) \quad \begin{cases} \int_0^{t_0} \int_G \delta T \, \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t + \int_G \delta T(\cdot, 0) \phi(\cdot, 0) \, \mathrm{d}x \\ \qquad \geq \sum_{i,j=1}^{m} \int_0^{t_0} \int_G a_{ij} \, \partial_{x_j} \delta T \, \partial_{x_i} \phi \, \mathrm{d}x \, \mathrm{d}t - \int_0^{t_0} \int_G \delta T \, \boldsymbol{\omega} \cdot \nabla \phi \, \mathrm{d}x \, \mathrm{d}t \end{cases}$$

with $\phi \in C^\infty_{0,+}$.
Summarizing Lemmata 1 and 2 we have

Theorem 1 *Let T be a solution of class* (20) *of the parabolic eq.* (19)*. Then, δT defined by* (30) *and* (22) *is of class $C^{0,1}(\overline{G} \times [0, t_0])$ and is, moreover, a weak subsolution* (*in the sense of* (32)) *of the "reduced equation"*

$$(33) \qquad \partial_t \delta T = \sum_{i,j=1}^{m} \partial_{x_i} (a_{ij} \, \partial_{x_j} \delta T) - \nabla_x \cdot (\boldsymbol{\omega} \, \delta T)$$

with $\boldsymbol{\omega}$ given in (31).

4 Decay in divergence-type parabolic equations

We consider in this section (sub-)solutions of the following initial-boundary-value problem

$$
\begin{cases} \partial_t h = \nabla \cdot (\lambda \nabla h - \boldsymbol{\omega}\, h) & \text{in } G \times (0,\infty), \\ h = 0 & \text{on } \partial G \times (0,\infty), \\ h(\cdot\,,0) = h_0 & \text{on } G \times \{t=0\}. \end{cases} \tag{34}
$$

Lortz et al. prove in [14] the following results:

Let $G \subset \mathbb{R}^m$ be a bounded, open set with ∂G of class $C^{2+\alpha}$. λ and $\boldsymbol{\omega}$ are assumed to be measurable and bounded on $G \times (0,\infty)$, in particular, there are constants $\lambda_0 > 0$, $M > 0$ such that

$$
\lambda \geq \lambda_0\,, \quad |\boldsymbol{\omega}| \leq M \qquad \text{in } G \times (0,\infty). \tag{35}
$$

In order to formulate an appropriate notion of weak solution consider for $t_0 > 0$ the spaces

$$
V_2 = \{ f \in L^2(G \times (0,t_0)) \,|\, \nabla f \text{ exists weakly and } \|f\|_{V_2} < \infty \}\,,
$$

$$
\overset{\circ}{V}_2 = \{ f \in V_2 \,|\, f(\cdot,t)|_{\partial G} = 0 \text{ in the trace sense for a.e. } t \in (0,t_0) \}
$$

with norm

$$
\|f\|_{V_2} := \sup_{(0,t_0)} \int_G |f|^2 \,\mathrm{d}x + \int_0^{t_0} \int_G |\nabla f|^2 \,\mathrm{d}x\,\mathrm{d}t\,.
$$

A function $h \in \overset{\circ}{V}_2$ is called a weak solution of (34) in $G \times (0,t_0)$ with initial value $h_0 \in L^2(G)$ if and only if h satisfies

$$
\int_0^{t_0} \int_G h\, \partial_t \phi \,\mathrm{d}x\,\mathrm{d}t + \int_G h_0\, \phi(\cdot,0) \,\mathrm{d}x = \int_0^{t_0} \int_G (\lambda \nabla h - \boldsymbol{\omega}\, h) \cdot \nabla \phi \,\mathrm{d}x\,\mathrm{d}t \tag{36}
$$

for any $\phi \in C_0^\infty(G \times (-\infty,t_0))$. h is called weak solution in $G \times (0,\infty)$ if and only if $h|_{G\times(0,t_0)}$ is a weak solution in $G \times (0,t_0)$ for any $t_0 > 0$. Solutions of this type decay (in time) in the following sense (cf. Theorem 1 in [14]):

Theorem 2 (Lortz et al.) *Let $h_0 \in L^\infty(G)$ and $\rho > 0$. There exist positive constants c_1, c_2, c_3, and d, depending on λ_0, M, G, and m, c_2 depending additionally on ρ, such that any weak solution h of* (34) *satisfies*

$$
\|h(\cdot,t)\|_{L^2(G)} \leq c_1 \,\|h_0\|_{L^2(G)}\, e^{-d\,t} \qquad \textit{in } (0,\infty), \tag{37}
$$

$$
\|h(\cdot,t)\|_{L^\infty(G)} \leq c_2 \,\|h_0\|_{L^2(G)}\, e^{-d\,t} \qquad \textit{in } (\rho,\infty), \tag{38}
$$

$$
\|h(\cdot,t)\|_{L^\infty(G)} \leq c_3 \,\|h_0\|_{L^\infty(G)}\, e^{-d\,t} \qquad \textit{in } (0,\infty). \tag{39}
$$

It is the aim of the present section to verify that Theorem 2 remains valid for *nonnegative* Lipschitz-continuous *sub*solutions of (34).

The proof in [14] relies mainly on the construction of a positive solution of an auxiliary problem to which Harnack-type inequalities apply. The basic methods used are nontrivial adaptations of Moser's iteration techniques to obtain, finally, pointwise estimates for solutions of (34) and the auxiliary problem. Large parts of the proof apply without changes to our situation and need not be repeated here. We discuss in the following only those points in the proof which require modifications and cite results about the (unchanged) auxiliary problem necessary for understanding the rest.

(i) Without any modification we can take over the results about weak solutions of the following auxiliary problem:

$$\begin{cases} \partial_t u = \nabla \cdot (\lambda \nabla u - \boldsymbol{\omega}\, u) & \text{in } G \times (0,\infty), \\ \mathbf{n} \cdot \nabla u = 0 & \text{on } \partial G \times (0,\infty), \\ u(\cdot\,, 0) = u_0 & \text{on } G \times \{t = 0\}. \end{cases} \tag{40}$$

λ, $\boldsymbol{\omega}$, and G are assumed to be as in problem (34), $\mathbf{n}$ means the exterior normal at ∂G, and $\boldsymbol{\omega}$ satisfies, additionally, $\mathbf{n} \cdot \boldsymbol{\omega}|_{\partial G \times (0,\infty)} = 0$. $u \in V_2$ is called weak solution in $G \times (0, t_0)$ with initial value $u_0 \in L^2(G)$ if and only if u satisfies (36) for any $\phi \in C_0^\infty(\mathbb{R}^m \times (-\infty, t_0))$. Weak solutions in $G \times (0,\infty)$ are defined as above.

Lortz et al. show that weak solutions of problem (36) (which are in fact locally Hölder continuous) with positive bounded initial values remain positive and bounded for all times, more precisely (cf. Theorem 2 in [14]):

There are positive constants c_4, c_5, depending only on λ_0, M, G, and m, such that the bound on u_0,

$$\underline{u}_0 \le u_0 \le \overline{u}_0 \qquad \text{on } G \tag{41}$$

with positive constants $\underline{u}_0$, $\overline{u}_0$, implies the bound

$$c_4\, \underline{u}_0 \le u \le c_5\, \overline{u}_0 \qquad \text{in } G \times (0,\infty)\,. \tag{42}$$

If, moreover, $\lambda, \boldsymbol{\omega} \in C^{1+\alpha,(1+\alpha)/2}(\overline{G} \times [0, t_0])$ and $u_0 \in C^{2+\alpha}(\overline{G})$ (and satisfying the compatibility conditions), then u is in fact a classical solution $\in C^{2+\alpha,(2+\alpha)/2}(\overline{G} \times [0, t_0])$.

(ii) The proof in [14] proceeds in two steps: Theorem 2 is proved first for classical solutions, and then extended to weak solutions via approximation by convex combinations of classical solutions. The last point requires uniqueness of the weak solution, which is obviously not given for *sub*solutions. Fortunately, the subsolution possesses in our case enough regularity to avoid the approximation argument. Concerning the auxiliary problem, λ is by assumption sufficiently smooth to allow classical solutions, but $\boldsymbol{\omega}$ is by definition (31) not better than (Lipschitz-)continuous; so we need here some approximation:

Let $t_0 > 0$ and $(\boldsymbol{\omega}_i) \subset C^\infty(\overline{G} \times [0, t_0])$ be an approximating sequence with $\mathbf{n} \cdot \boldsymbol{\omega}_i|_{\partial G \times [0,t_0]} = 0$, $\|\boldsymbol{\omega}_i\|_{C^0} \le M$, $i \in \mathbb{N}$, and

$$\|\boldsymbol{\omega}_i - \boldsymbol{\omega}\|_{C^0} \to 0 \qquad \text{for } i \to \infty. \tag{43}$$

Let (u_i) be the associated sequence of classical solutions of (40) with $\lambda \in C^{1+\alpha,(1+\alpha)/2}(\overline{B_R} \times [0, t_0])$ and $u_0 \in C^{2+\alpha}(\overline{G})$ satisfying (42) uniformly with respect to i. According to the estimate (5.3) in [14] u_i is, moreover, uniformly bounded in V_2:

$$\|u_i\|_{V_2} < K = K(\lambda_0, M, G, m; u_0, t_0)\,, \quad i \in \mathbb{N}. \tag{44}$$

Now we prove (37). Multiplying $(40)_1$ with u_i and $\boldsymbol{\omega}_i$ by $(h/u_i)^2$ and integrating over G one obtains after integrating by parts

$$\int_G \partial_t u_i (h/u_i)^2 \mathrm{d}x = -2\int_G (h/u_i)(\lambda\nabla u_i - \boldsymbol{\omega}_i u_i)\cdot\nabla(h/u_i)\,\mathrm{d}x \tag{45}$$

for a.e. $t \in (0, t_0)$. Note that (45) makes sense for Lipschitz-continuous h when observing the correspondence $C^{0,1}(\overline{B_R}\times[0,t_0]) \sim W^{1,\infty}(G\times(0,t_0))$. h itself satisfies (36) as inequality for any $\phi \in C^\infty_{0,+}(G\times(-\infty,t_0))$. Choosing for fixed $i \in \mathbb{N}$ a sequence of test functions $(\phi_{ij})_{j\in\mathbb{N}}$ of the form $\phi_{ij}(\mathbf{x},t) = \psi_{ij}(\mathbf{x},t)\,\chi(t)$ with $\|\psi_{ij} - h/u_i\|_{\mathcal{H}} \to 0$ for $j \to \infty$ and $\chi \in C^\infty_{0,+}((-\infty,t_0))$ one obtains after integration by parts (in t):

$$\int_0^{t_0}\int_G \partial_t h\,\psi_{ij}\,\chi\,\mathrm{d}x\mathrm{d}t \le -\int_0^{t_0}\int_G (\lambda\nabla h - \boldsymbol{\omega} h)\cdot\nabla\psi_{ij}\,\chi\,\mathrm{d}x\mathrm{d}t.$$

Here $\|\cdot\|_{\mathcal{H}}$ denotes the Hilbert space norm

$$\|f\|^2_{\mathcal{H}} := \int_0^{t_0}\int_G \big(|f|^2 + |\nabla f|^2\big)\mathrm{d}x\mathrm{d}t.$$

Thus, in the limit $j \to \infty$, one obtains

$$\int_0^{t_0}\Big(\int_G \partial_t h\,h/u_i\,\mathrm{d}x\Big)\chi\,\mathrm{d}t \le -\int_0^{t_0}\Big(\int_G (\lambda\nabla h - \boldsymbol{\omega} h)\cdot\nabla(h/u_i)\,\mathrm{d}x\Big)\chi\,\mathrm{d}t,$$

and, furthermore, since $\chi \in C^\infty_{0,+}((-\infty,t_0))$ being arbitrary,

$$\int_G \partial_t h\,h/u_i\,\mathrm{d}x \le -\int_G (\lambda\nabla h - \boldsymbol{\omega} h)\cdot\nabla(h/u_i)\,\mathrm{d}x \tag{46}$$

for a.e. $t \in (0, t_0)$. Combining (45) and (46) we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_G h^2/u_i\,\mathrm{d}x &= \int_G \big(2\,\partial_t h\,h/u_i - \partial_t u_i\,(h/u_i)^2\big)\,\mathrm{d}x\\ &\le -2\int_G \lambda\,u_i|\nabla(h/u_i)|^2\mathrm{d}x + 2\int_G h(\boldsymbol{\omega} - \boldsymbol{\omega}_i)\cdot\nabla(h/u_i)\,\mathrm{d}x.\end{aligned} \tag{47}$$

Denoting $\|\boldsymbol{\omega} - \boldsymbol{\omega}_i\|_{C^0}$ by ϵ_i, Poincaré's constant in G with zero boundary conditions by C_G, and observing (35), (42), and (44) the right-hand side in (47) can be estimated by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_G h^2/u_i\,\mathrm{d}x &\le -2\,\lambda_0\,c_4\,\underline{u_0}\int_G |\nabla(h/u_i)|^2\mathrm{d}x + 2\|\boldsymbol{\omega} - \boldsymbol{\omega}_i\|_{C^0}\int_G h|\nabla(h/u_i)|\,\mathrm{d}x\\ &\le -2\,\frac{\lambda_0}{C_G^2}\,\frac{c_4\,\underline{u_0}}{c_5\,\overline{\overline{u_0}}}\int_G h^2/u_i\,\mathrm{d}x + 2\,\epsilon_i\int_G h\,|(\nabla h/u_i) - (h/u_i^2)\nabla u_i|\,\mathrm{d}x\\ &=: -K_1\int_G h^2/u_i\,\mathrm{d}x + K_2\,\epsilon_i\end{aligned}$$

with positive constants $K_1 = K_1(\lambda_0, M, G, m; \underline{u_0}/\overline{u_0})$ and $K_2 = K_2(\lambda_0, M, G, m; u_0, h, t_0)$. Applying Gronwall's inequality yields

$$\int_G h^2/u_i \,\mathrm{d}x \le \int_G h_0^2/u_0 \,\mathrm{d}x \;\mathrm{e}^{-K_1 t} + \epsilon_i\, K_2/K_1.$$

Taking now the limit $i \to \infty$ and applying once more (42) yields, finally,

$$\int_G h^2 \,\mathrm{d}x \le \frac{c_5\, \overline{u_0}}{c_4\, \underline{u_0}} \int_G h_0^2 \,\mathrm{d}x \;\mathrm{e}^{-K_1 t},$$

which is (37). Note that we are free to choose $u_0 = const > 0$ and hence $\overline{u_0} = \underline{u_0}$, and that K_1 does not depend on t_0.

(iii) The proofs of the inequalities (38) and (39) are based on (37) and the following estimates (cf. Theorem 3 in [14]):

Let ρ, t, and s such that $0 < \rho < t$, $\rho \le 1/2$, and $0 < \rho \le 1$. There exist then positive constants c_6 and c_7 such that

$$\|h\|_{L^\infty(G\times(t,t+s))} \le c_6\, \|h\|_{L^2(G\times(t-\rho,t+s+\rho))}, \tag{48}$$

$$\|h\|^2_{L^\infty(G\times(0,t))} \le c_7\, \big(\|h\|^2_{L^2(G\times(0,t))} + \|h_0\|^2_{L^\infty(G)}\big) \tag{49}$$

for any classical solution h of problem (34). c_6, c_7 depend on λ_0, M, G, and m; c_6 depends, additionaly, on ρ and c_7 on t.

The proof proceeds via a series of partly rather tricky estimates, which apply without changes to our situation as well. The only difference is in the "starting" equation (2.1a) leading to inequality (3.1) in [14]. Instead, our starting point is again (36) as inequality. Choosing a sequence of test functions $(\phi_i) \subset C^\infty_{0,+}(G \times (-\infty, t_0))$ of the form $\phi_i(\mathbf{x}, t) = \psi_i(\mathbf{x}, t)\, \chi_i(t)$ with $\|\psi_i - h^{\gamma-1}\|_{\mathcal{H}} \to 0$, $\|\chi_i - \chi\|_{L^2} \to 0$ for $i \to \infty$ one obtains after integration by parts (in t) and in the limit $i \to \infty$:

$$\int_{t_1}^{t_2} \Big(\int_G \partial_t h\, h^{\gamma-1} \,\mathrm{d}x\Big)\, \chi \,\mathrm{d}t \le -\int_{t_1}^{t_2} \Big(\int_G (\lambda \nabla h - \boldsymbol{\omega} h) \cdot \nabla h^{\gamma-1} \,\mathrm{d}x\Big)\, \chi \,\mathrm{d}t. \tag{50}$$

$\chi \ge 0$ is here some differentiable function in $(t_1, t_2) \subset [0, t_0]$ vanishing outside. Note that only the case $\gamma \ge 2$ is relevant here. Rearranging (50) and using (35) yields

$$\begin{aligned}\frac{1}{\gamma(\gamma-1)} \int_{t_1}^{t_2} \partial_t\Big(\int_G h^\gamma \,\mathrm{d}x\Big)\, \chi \,\mathrm{d}t + \int_{t_1}^{t_2} \Big(\int_G \lambda\, h^{\gamma-2} |\nabla h|^2 \mathrm{d}x\Big)\, \chi \,\mathrm{d}t \\ \le M \int_{t_1}^{t_2} \Big(\int_G h^{\gamma-1} |\nabla h| \,\mathrm{d}x\Big)\, \chi \,\mathrm{d}t\,.\end{aligned}$$

Integration by parts on the left-hand side and using the estimate (3.3) in [14] on the right-hand side yields then

$$\begin{aligned}\frac{1}{\gamma(\gamma-1)} \Big[\int_G h^\gamma \,\mathrm{d}x\, \chi\Big]_{t_1}^{t_2} + \frac{3}{4}\lambda_0 \int_{t_1}^{t_2} \Big(\int_G h^{\gamma-2} |\nabla h|^2 \mathrm{d}x\Big)\, \chi \,\mathrm{d}t \\ \le \int_{t_1}^{t_2} \int_G h^\gamma \,\mathrm{d}x \Big(\frac{|\chi'|}{\gamma(\gamma-1)} + \frac{M^2}{\lambda_0}\, \chi\Big) \mathrm{d}t,\end{aligned}$$

which is (3.1) in [14]. Note, finally, that the technical restriction $h > 0$ with subsequent relaxation is not necessary here, since negative powers of h and $\log h$-estimates do not appear.

5 Decay of purely toroidal dynamo fields

In this section we apply the foregoing results to purely toroidal dynamo fields in a plane layer $L = \mathbb{R}^2 \times (0,l)$ and in a ball B_R. In the former case we assume the toroidal scalar T to be a smooth, i.e. of class $C^{2+\alpha,(2+\alpha)/2}(\overline{L} \times [0,\infty))$, $\mathcal{P}$-periodic solution of the following problem (cf. (8)):

$$(51) \quad \begin{cases} \partial_t T = \partial_z(\lambda\,\partial_z T) + \lambda \nabla_h \cdot \nabla_h T - \partial_z W_T - \mathbf{v}\cdot\nabla_h T & \\ \qquad +\langle \partial_z W_T + \mathbf{v}\cdot\nabla_h T\rangle & \text{in } \mathbb{R}^2 \times [0,l] \times (0,\infty), \\ T \quad \mathcal{P}-\text{periodic} & \text{in } \mathbb{R}^2 \times [0,l] \times (0,\infty), \\ T = 0 & \text{on } \mathbb{R}^2 \times \{z=0,l\} \times (0,\infty), \\ T(\cdot\,,0) = T_0\,, \quad \langle T_0\rangle = 0 & \text{on } \mathbb{R}^2 \times [0,l] \times \{t=0\}. \end{cases}$$

Replacing $\mathbb{R}^2$ by the bounded region $\mathcal{P}$, eq. $(51)_1$ fits into the framework of sec. 3. According to Theorem 1 the quantity $\delta T = \max_{\mathcal{P}} T - \min_{\mathcal{P}} T$ is of class $C^{0,1}([0,l] \times [0,t_0])$ and is, furthermore, a weak subsolution of the problem

$$\begin{cases} \partial_t \delta T = \partial_z(\lambda\,\partial_z \delta T) - \partial_z(\omega\,\delta T) & \text{in } (0,l) \times (0,t_0), \\ \delta T = 0 & \text{on } \{z=0,l\} \times (0,t_0), \\ \delta T(\cdot\,,0) = \delta T_0 & \text{on } (0,l) \times \{t=0\} \end{cases}$$

for any $t_0 > 0$. ω is here determined according to (31) by W and T, and $\delta T_0 := \max_{\mathcal{P}} T_0 - \min_{\mathcal{P}} T_0$. Note that $\lambda \in C^{1+\alpha,(1+\alpha)/2}(\overline{L} \times [0,t_0])$ and is horizontally symmetric by assumption, whereas ω is merely bounded and continuous by definition. δT is nonnegative due to the zero mean condition $\langle T\rangle = 0$. Recall, furthermore, the correspondence $C^{0,1}([0,l] \times [0,t_0]) \sim W^{1,\infty}((0,l) \times (0,t_0))$, which implies δT being also a weak (sub)solution in the sense of Theorem 2, and observe the equivalence of norms:

$$\max_L T \le \max_{[0,l]} \delta T \le 2 \max_L T\,.$$

Therefore, Theorem 2 yields the result:

Theorem 3 *Any smooth, $\mathcal{P}$-periodic solution of problem* (51) *decays in time according to*

$$\max_L |T(\cdot,t)| \le C\, \max_L |T_0|\, e^{-d\,t}, \qquad t \ge 0\,.$$

C and d depend only on l, λ_0, and a pointwise bound on $\partial_\tau W$ (or v_z).

In the spherical case the modified toroidal scalar $\mathcal{T} = rT$ is governed by (cf. (16)):

$$(52) \quad \begin{cases} \partial_t \mathcal{T} = \partial_r(\lambda\,\partial_r \mathcal{T}) + \lambda \nabla_{nr} \cdot \nabla_{nr} \mathcal{T} - \partial_r W_{\mathcal{T}} & \\ \qquad -\mathbf{v}\cdot\nabla_{nr}\mathcal{T} + \langle \partial_r W_{\mathcal{T}} + \mathbf{v}\cdot\nabla_{nr}\mathcal{T}\rangle & \text{in } B_R \times (0,\infty), \\ \mathcal{T} = 0 & \text{on } (S_R \cup \{r=0\}) \times (0,\infty), \\ \mathcal{T}(\cdot\,,0) = \mathcal{T}_0\,, \quad \langle \mathcal{T}_0\rangle = 0 & \text{on } B_R \times \{t=0\}\,. \end{cases}$$

Note that $(52)_1$ does not fit precisely into the framework of sec. 3, since B_R is a "warped" product of its factors $[0,R]$ and S_r. In particular, the nonradial gradient ∇_{nr} depends also on r. However, checking the proofs in sec. 3 one makes sure that Theorem 1 applies to this situation as well. Thus, starting with a smooth solution of (52) Theorem 1 yields $\delta\mathcal{T}(r,t) = \max_{S_r}\mathcal{T}(\cdot,t) - \min_{S_r}\mathcal{T}(\cdot,t)$ to be of class $C^{0,1}([0,R]\times[0,t_0])$ and to be a weak subsolution of the problem

$$\begin{cases} \partial_t\delta\mathcal{T} = \partial_r(\lambda\,\partial_r\delta\mathcal{T}) - \partial_r(\omega\,\delta\mathcal{T}) & \text{in } (0,R)\times(0,t_0), \\ \delta\mathcal{T} = 0 & \text{on } \{r=0,R\}\times(0,t_0), \\ \delta\mathcal{T}(\,\cdot\,,0) = \delta\mathcal{T}_0 & \text{on } (0,R)\times\{t=0\} \end{cases}$$

for any $t_0 > 0$. Proceeding as in the plane case, Theorem 2 yields then the result:

Theorem 4 *Any smooth solution of problem* (52) *decays in time according to*

$$\max_{B_R}|\mathcal{T}(\cdot,t)| \le C\,\max_{B_R}|\mathcal{T}_0|\,e^{-d\,t}, \qquad t\ge 0\,. \tag{53}$$

C and d depend only on R, λ_0, and a pointwise bound on $\partial_\tau W$ (or v_r).

Note that (53), if expressed in terms of the original variable T, amounts to the nonuniform bound

$$\max_{S_r}|T(\cdot,t)| \le C\,\frac{R}{r}\,\max_{B_R}|T_0|\,e^{-d\,t}, \qquad 0<r\le R\,, \quad t\ge 0\,.$$

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References

[1] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, England 1978).

[2] E. C. Bullard and H. Gellman, "Homogeneous dynamos and terrestrial magnetism", *Phil. Trans. R. Soc. Lond.* A**247**, 213–278 (1954).

[3] W. M. Elsasser, "Hydromagnetic dynamo theory", *Rev. Mod. Phys.* **28**, 135–163 (1956).

[4] S. Childress, *Théorie Magnétohydrodynamique de l'Effet Dynamo*, Report of Département Mécanique de la Faculté des Sciences, Paris (1969).

[5] F. H. Busse, "Mathematical problems of dynamo theory", in: *Applications of Bifurcation Theory*, (Ed. P. H. Rabinowitz) pp. 175–202 (Academic Press: New York, San Francisco, London 1977).

[6] D. J. Ivers and R. W. James, "An antidynamo theorem for partly symmetric flows", *Geophys. Astrophys. Fluid Dynam.* **44**, 271–278 (1988).

[7] D. J. Ivers, *Antidynamo theorems* (Ph.D. Thesis, University of Sydney 1984).

[8] T. O. Cowling, "The magnetic field of sunspots", *Mon. Not. R. astr. Soc.* **94**, 39–48 (1934).

[9] W. M. Elsasser, “Induction effects in terrestrial magnetism. I.Theory”, *Phys. Rev.* **69**, 106–116 (1946).

[10] R. Kaiser, B. J. Schmitt, and F. H. Busse, “On the invisible dynamo”, *Geophys. Astrophys. Fluid Dynam.* **77**, 93–109 (1994).

[11] D. J. Ivers and R. W. James, “Axisymmetric antidynamo theorems in compressible non-uniform conducting fluids”, *Phil. Trans. R. Soc. Lond.* A**312**, 179–218 (1984).

[12] G. E. Backus, “A class of self-sustaining dissipative spherical dynamos”, *Ann. Phys.* **4**, 372–447 (1958).

[13] R. Kaiser, “The non-radial velocity theorem revisited”, *Geophys. Astrophys. Fluid Dynam.* **101**, 185–197 (2007).

[14] D. Lortz, R. Meyer-Spasche, and E. W. Stredulinsky, “Asymptotic Behavior of the Solutions of Certain Parabolic Equations”, *Comm. Pure Appl. Math.* **37**, 677–703 (1984).

[15] B. J. Schmitt and W. von Wahl, “Decomposition of solenoidal fields into poloidal fields, toroidal fields, and the mean flow. Applications to the Boussinesq-equations”, in: *The Navier Stokes Equations II – Theory and Numerical Methods*, Lecture Notes in Mathematics **1530**, (Ed. I. G. Heywood, K. Masuda, R. Rautmann, and S. A. Solonnikov) pp. 291-305 (Springer-Verlag, Berlin, Heidelberg, New York 1992).

[16] B. J. Schmitt, “The poloidal-toroidal representation of solenoidal fields in spherical domains”, *Analysis* **15**, 257–277 (1995).

[17] B. J. Schmitt, “Purely toroidal solutions of the kinematic dynamo equation”, *Nonlinear differ. equ. appl. (NoDEA)* **4**, 217–231 (1997).

[18] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs Vol. **23** (American Mathematical Society, Providence, R. I. 1968).

[19] B. J. Schmitt, *Abschätzungen für die Poloidal-Toroidal-Zerlegung von Solenoidalen Vektorfeldern in Dreidimensionalen Kugeln, Anwendungen auf Toroidale Lösungen der Kinematischen Dynamogleichung*, p. 63 (Thesis, University of Bayreuth 1994).