

Flat Galaxies with Dark Matter Halos—Existence and Stability

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Abstract: We consider a model for a flat, disk-like galaxy surrounded by a halo of dark matter, namely a Vlasov-Poisson type system with two particle species, the stars which are restricted to the galactic plane and the dark matter particles. These constituents interact only through the gravitational potential which stars and dark matter create collectively. Using a variational approach we prove the existence of steady state solutions and their nonlinear stability under suitably restricted perturbations.

1. Introduction

Around 1970 astrophysicists noticed that in typical spiral galaxies the rotation velocities of the stars, when computed in the gravitational potential of the visible matter, do not fit with their observed rotation velocities. It was then conjectured that such galaxies are surrounded by a halo of so far not directly observable dark matter in such a way that the rotation velocities of the stars are consistent with the resulting gravitational potential [11]. For an introduction to dark matter we refer to [3, Chap. 10] and the references there.

The distribution of the stars in a galaxy is usually modeled by a density function on phase space, and it is assumed that collisions are sufficiently rare to be neglected and that the stars interact only by the gravitational potential which they create collectively. In a non-relativistic setting this results in a system of partial differential equations which in the mathematics literature is known as the Vlasov-Poisson system, cf. [27]. While the true physical nature (and existence) of dark matter are still conjectural, we are aware of at least one astrophysics investigation where it is also modeled as Vlasov-type matter, cf. [28]. Given the fact that the only role which galactic dark matter has to play is to provide the mass and hence the gravitational potential needed to resolve the discrepancy concerning the rotation velocities of the stars, such a description of dark matter seems natural.

In the present paper we investigate a model for a flat, disk-like galaxy with a halo of dark matter where both the distribution of the stars in the galactic plane and the distribution of the dark matter particles in the halo obey a Vlasov equation, and the interaction among stars, dark matter, and between these two constituents is through the gravitational potential which all the particles (stars and dark matter) create collectively.

Following the practice in astrophysics we assume that the stars are restricted to a plane which we take to be the x_1, x_2 plane. Their distribution on phase space is given by $\tilde{f} = \tilde{f}(t, \tilde{x}, \tilde{v}) \geq 0$, where $t \geq 0$ denotes time and $\tilde{x}, \tilde{v} \in \mathbb{R}^2$ denote position and velocity in the galactic plane. The distribution of the dark matter particles is given by $f = f(t, x, v) \geq 0$, where $x, v \in \mathbb{R}^3$ denote position and velocity in three dimensional space. The evolution of the galaxy and its halo is then governed by the following Vlasov-Poisson type system of equations:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U_e \cdot \nabla_v f = 0, \quad (1.1)$$

$$\partial_t \tilde{f} + \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{f} - \nabla_{\tilde{x}} U_e(\cdot, 0) \cdot \nabla_{\tilde{v}} \tilde{f} = 0, \quad (1.2)$$

$$U_e(t, x) = U(t, x) + \tilde{U}(t, x) = - \int_{\mathbb{R}^3} \frac{\rho(t, y)}{|x-y|} dy - \int_{\mathbb{R}^2} \frac{\tilde{\rho}(t, \tilde{y})}{|x-(\tilde{y}, 0)|} d\tilde{y}, \quad (1.3)$$

$$\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv, \quad \tilde{\rho}(t, \tilde{x}) = \int_{\mathbb{R}^2} \tilde{f}(t, \tilde{x}, \tilde{v}) d\tilde{v}. \quad (1.4)$$

Here ρ and $\tilde{\rho}$ are the spatial mass densities of dark matter respectively stars, U and \tilde{U} are the induced Newtonian potentials, and U_e denotes the potential of the system as a whole, i.e., the effective potential which determines the particle orbits. In order that the stars remain in their plane it is sufficient to require that $f(t, \tilde{x}, x_3, \tilde{v}, v_3) = f(t, \tilde{x}, -x_3, \tilde{v}, -v_3)$, a condition which at least formally is preserved by solutions of the system and which implies that $\nabla U(t, \tilde{x}, 0)$ is parallel to the plane; for $\nabla \tilde{U}(t, \tilde{x})$ this is true automatically. Throughout this paper we use the convention that variables with (without) tilde denote flat (non-flat) quantities.

To our knowledge a fully non-linear model where the gravitational interaction within both types of matter and between the two types is taken into account has so far not been investigated. Our aim is to prove the existence and non-linear stability of steady state configurations to this system. We obtain such stable steady states as minimizers of the total energy

$$\begin{aligned} & \frac{1}{2} \iint |v|^2 f dx dv + \frac{1}{2} \iint |\tilde{v}|^2 \tilde{f} d\tilde{x} d\tilde{v} \\ & + \frac{1}{2} \int U_e(x) \rho(x) dx + \frac{1}{2} \int U_e(\tilde{x}, 0) \tilde{\rho}(\tilde{x}) d\tilde{x}, \end{aligned}$$

satisfying suitable constraints. This so-called energy-Casimir approach was developed for the usual, three dimensional Vlasov-Poisson system, i.e., $\tilde{f} = 0$ in the above, in [12–16, 25], see also [7, 19, 27, 29]. The approach has also been used to prove the existence of stable steady states for flat galaxies without a halo, i.e., with $f = 0$ in the above, cf. [9, 10, 24]. The fact that in the present situation the energy is a functional acting on two functions together with the potential interaction terms between the flat and the non-flat component requires substantial new ingredients in the basic scheme. One pitfall to avoid is that for a minimizer of the above energy functional one of the two components might vanish.

Besides the above stability results it is known that global classical solutions to the initial value problem for the usual three dimensional Vlasov-Poisson system exist,

cf. [21, 22], while local classical and global weak solutions exist in the flat case without halo, cf. [6]. For the situation at hand nothing is known about the initial value problem, but we conjecture that the analogue of [6] for weak solutions remains true. Our stability result is conditional in the sense that it holds for solutions as long as they exist and preserve the required conserved quantities. For more information on the Vlasov-Poisson system in general we refer to the review article [27].

The paper proceeds as follows. In the next section we formulate our variational problem and our main result on the existence of minimizers. In Sect. 3 we establish properties of the potentials which allow us to define and control the potential energies, in particular the interaction terms. Next we collect some relevant results about the decoupled variational problems where one of the two components is missing; these facts are established in an Appendix. In Sect. 5 we show that the energy functional is bounded from below, that not all the mass can escape to infinity along a minimizing sequence, and we investigate the splitting properties of the functional. With these prerequisites we can then prove the existence of minimizers in Sect. 6. The fact that such minimizers are steady states together with some of their properties are established in Sect. 7. In Sect. 8 we finally investigate the stability estimate resulting from their minimizing property.

2. Variational Setup

We denote the set of non-negative, Lebesgue integrable functions by $L_+^1(\mathbb{R}^n)$. For $f \in L_+^1(\mathbb{R}^6)$ and $\rho \in L_+^1(\mathbb{R}^3)$ we denote by

$$\rho_f(x) := \int_{\mathbb{R}^3} f(t, x, v) dv, \quad U_\rho(x) := - \int_{\mathbb{R}^3} \frac{\rho(y)}{|x - y|} dy$$

the induced spatial density and gravitational potential; we write $U_f = U_{\rho_f}$. Similarly, for $\tilde{f} \in L_+^1(\mathbb{R}^4)$ and $\tilde{\rho} \in L_+^1(\mathbb{R}^2)$,

$$\rho_{\tilde{f}}(\tilde{x}) := \int_{\mathbb{R}^2} \tilde{f}(t, \tilde{x}, \tilde{v}) d\tilde{v}, \quad U_{\tilde{\rho}}(\tilde{x}) := - \int_{\mathbb{R}^2} \frac{\tilde{\rho}(\tilde{y})}{|\tilde{x} - (\tilde{y}, 0)|} d\tilde{y},$$

and to abbreviate we sometimes write $\tilde{\rho}$ and \tilde{U} instead of $\rho_{\tilde{f}}$ and $U_{\tilde{\rho}}$; notice that the latter is defined on \mathbb{R}^3 . In what follows we do not explicitly denote the domain of integration— \mathbb{R}^3 or \mathbb{R}^2 —unless in cases of ambiguity. The integrability properties of these potentials are investigated in Sect. 3. Next we define the various parts of the energy functional. For $f \in L_+^1(\mathbb{R}^6)$ and $\tilde{f} \in L_+^1(\mathbb{R}^4)$,

$$E_{\text{kin}}(f) := \frac{1}{2} \iint |v|^2 f(x, v) dv dx, \quad E_{\text{kin}}(\tilde{f}) := \frac{1}{2} \iint |\tilde{v}|^2 \tilde{f}(\tilde{x}, \tilde{v}) d\tilde{v} d\tilde{x},$$

$$E_{\text{pot}}(f) := - \frac{1}{2} \iint \frac{\rho_f(x) \rho_f(y)}{|x - y|} dx dy, \quad E_{\text{pot}}(\tilde{f}) := - \frac{1}{2} \iint \frac{\tilde{\rho}_{\tilde{f}}(\tilde{x}) \tilde{\rho}_{\tilde{f}}(\tilde{y})}{|\tilde{x} - \tilde{y}|} d\tilde{x} d\tilde{y},$$

denote the kinetic and potential energies of the non-flat and flat components. The total energy of each component is then defined by

$$\mathcal{H}(f) := E_{\text{kin}}(f) + E_{\text{pot}}(f), \quad \mathcal{H}(\tilde{f}) := E_{\text{kin}}(\tilde{f}) + E_{\text{pot}}(\tilde{f}).$$

Finally,

$$\begin{aligned} \mathcal{H}(f, \tilde{f}) &= \mathcal{H}(f) + \mathcal{H}(\tilde{f}) + \frac{1}{2} \int U_{\tilde{f}}(x) \rho_f(x) dx + \frac{1}{2} \int U_f(\tilde{x}, 0) \rho_{\tilde{f}}(\tilde{x}) d\tilde{x}, \\ &= \mathcal{H}(f) + \mathcal{H}(\tilde{f}) + \int U_{\tilde{f}}(x) \rho_f(x) dx \end{aligned}$$

is the total energy of the state (f, \tilde{f}) . In Sect. 3, where we investigate the existence of all these integrals on the constraint set defined below, we will also see that the two interaction terms are equal.

We wish to minimize this functional over the constraint set

$$\begin{aligned} \mathcal{F}_{\mathbf{M}} := \left\{ (f, \tilde{f}) \mid f \in L^1_+(\mathbb{R}^6), \tilde{f} \in L^1_+(\mathbb{R}^4), \|f\|_1 \leq M, \|f\|_{1+1/k} \leq N, \right. \\ \left. \|\tilde{f}\|_1 \leq \tilde{M}, \|\tilde{f}\|_{1+1/\tilde{k}} \leq \tilde{N}, E_{\text{kin}}(f) + E_{\text{kin}}(\tilde{f}) < \infty, \right. \\ \left. f(\tilde{x}, x_3, \tilde{v}, v_3) = f(\tilde{x}, -x_3, \tilde{v}, -v_3) \right\}, \end{aligned}$$

where $\mathbf{M} := (M, N, \tilde{M}, \tilde{N})$ denotes the constraint vector whose components are all strictly positive, $\|\cdot\|_p$ denotes the usual L^p norm, and

$$0 < k < 7/2, \quad 0 < \tilde{k} < 2.$$

In Sect. 3 and 5 we will see that the total energy functional is well defined and bounded from below on this set. The following theorem is our main result.

Theorem 2.1. *Let $(f_j, \tilde{f}_j) \subset \mathcal{F}_{\mathbf{M}}$ be a minimizing sequence of \mathcal{H} . Then there exists $(f_0, \tilde{f}_0) \in \mathcal{F}_{\mathbf{M}}$, a subsequence again denoted by (f_j, \tilde{f}_j) and a sequence of shift vectors $(\tilde{a}_j) \subset \mathbb{R}^2$ such that with $T_j f_j(x, v) := f_j(x + (\tilde{a}_j, 0), v)$, $T_j \tilde{f}_j(\tilde{x}, \tilde{v}) := \tilde{f}_j(\tilde{x} + \tilde{a}_j, \tilde{v})$,*

$$\begin{aligned} T_j f_j \rightharpoonup f_0, \quad T_j \tilde{f}_j \rightharpoonup \tilde{f}_0 \text{ weakly in } L^{1+1/k}(\mathbb{R}^6) \text{ or } L^{1+1/\tilde{k}}(\mathbb{R}^4) \text{ respectively,} \\ E_{\text{pot}}(T_j f_j - f_0) \rightarrow 0, \quad E_{\text{pot}}(T_j \tilde{f}_j - \tilde{f}_0) \rightarrow 0, \end{aligned}$$

and

$$\int (\rho_{T_j f_j} - \rho_{f_0}) U_{T_j \tilde{f}_j - \tilde{f}_0} dx \rightarrow 0.$$

Moreover (f_0, \tilde{f}_0) is a minimizer of \mathcal{H} over $\mathcal{F}_{\mathbf{M}}$.

The spatial shifts parallel to the (x_1, x_2) plane are necessary due to the invariance of the total energy and the constraint set under such shifts. If (f_0, \tilde{f}_0) is a minimizer of \mathcal{H} , then $(T_j f_0, T_j \tilde{f}_0)$ is a minimizing sequence for any choice of shift vectors $\tilde{a}_j \in \mathbb{R}^2$ which is weakly convergent to a minimizer only if we shift our frame of reference accordingly.

The constraints on $\|f\|_{1+1/k}$ and $\|\tilde{f}\|_{1+1/\tilde{k}}$ in the definition of the set $\mathcal{F}_{\mathbf{M}}$ play the role of the Casimir constraints, and it does not seem to be possible to include these Casimirs into the functional to be minimized, as was done for example in [14] for the purely three dimensional and in [24] for the purely flat problem. In the latter cases these Casimir functionals can be replaced by more general ones of the form $\iint \Phi(f(x, v)) dv dx$ with some suitable prescribed function Φ . The Casimir constraint determines the microscopic equation of state of the resulting steady states, and the choice in the present paper

restricts these steady states to the so-called polytropic case, cf. Thm. 7.2. In astrophysics polytropic states have been and are studied extensively, also in the context of dark matter, cf. the discussion in [5, 8, 18, 23, 31] and the references there. However, from the applications point of view it is desirable to extend the present analysis to non-polytropic and possibly non-isotropic states along the lines in [14–16]. Such an extension does not seem straightforward to the authors since the form of the constraints and in particular their scaling properties play an important role.

3. Preliminaries

We start by collecting some well known estimates for the spatial densities and potential energies induced by elements from the constraint set $\mathcal{F}_{\mathbf{M}}$.

Lemma 3.1. *Let $(f, \tilde{f}) \in \mathcal{F}_{\mathbf{M}}$ and define $n := k + 3/2$, $\tilde{n} := \tilde{k} + 1$. Then $\rho_f \in L^{1+1/n}(\mathbb{R}^3)$, $\rho_{\tilde{f}} \in L^{1+1/\tilde{n}}(\mathbb{R}^2)$ with*

$$\begin{aligned} \|\rho_f\|_{1+1/n} &\leq C N^{(k+1)/(n+1)} E_{\text{kin}}(f)^{3/(2k+5)}, \\ \|\rho_{\tilde{f}}\|_{1+1/\tilde{n}} &\leq C \tilde{N}^{(\tilde{k}+1)/(\tilde{n}+1)} E_{\text{kin}}(\tilde{f})^{1/(\tilde{k}+2)}, \end{aligned}$$

and

$$\begin{aligned} -E_{\text{pot}}(f) &\leq C \|\rho_f\|_{6/5}^2 \leq C_{\mathbf{M}} E_{\text{kin}}(f)^{1/2}, \\ -E_{\text{pot}}(\tilde{f}) &\leq C \|\tilde{\rho}_{\tilde{f}}\|_{4/3}^2 \leq C_{\mathbf{M}} E_{\text{kin}}(\tilde{f})^{1/2}, \end{aligned}$$

where the constant $C > 0$ is universal and $C_{\mathbf{M}} > 0$ depends on the constraint vector \mathbf{M} . By the restrictions on k and \tilde{k} , $1 + 1/n > 6/5$ and $1 + 1/\tilde{n} > 4/3$ so that $\rho_f \in L^{6/5}(\mathbb{R}^3)$, $\rho_{\tilde{f}} \in L^{4/3}(\mathbb{R}^2)$.

Proof. Given $R > 0$ we split the v -integral and use Hölder’s inequality and the definition of the kinetic energy to find that

$$\begin{aligned} \rho_f(x) &= \int_{|v| \leq R} f(x, v) dv + \int_{|v| > R} f(x, v) dv \\ &\leq \left(\frac{4\pi}{3} R^3\right)^{1/(k+1)} \left(\int f^{1+1/k}(x, v) dv\right)^{k/(k+1)} + \frac{1}{R^2} \int |v|^2 f(x, v) dv. \end{aligned}$$

We optimize this estimate in R , take the resulting estimate for $\rho_f(x)$ to the power $1 + 1/n$ and integrate with respect to x to obtain the estimate for ρ_f . The estimate for $\rho_{\tilde{f}}$ follows the same lines. The last two inequalities follow by interpolation and the Hardy-Littlewood-Sobolev inequality. \square

In order to analyze the mixed term in $\mathcal{H}(f, \tilde{f})$ we need some information on the integrability of the flat potential in \mathbb{R}^3 .

Lemma 3.2. *Let $\tilde{\rho} \in L^{4/3}(\mathbb{R}^2)$. Then $U_{\tilde{\rho}} \in L^6(\mathbb{R}^3)$ and*

$$\|U_{\tilde{\rho}}\|_{L^6(\mathbb{R}^3)} \leq C \|\tilde{\rho}\|_{L^{4/3}(\mathbb{R}^2)}.$$

Proof. We use the general form of the Minkowski inequality, cf. [20, 2.4], and the weak Young inequality to obtain

$$\begin{aligned} \|U_{\tilde{\rho}}\|_6^6 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} \frac{\tilde{\rho}(\tilde{y})}{|x - (\tilde{y}, 0)|} d\tilde{y} \right)^6 dx_3 d\tilde{x} \\ &\leq \int \left[\int \left(\int \frac{\tilde{\rho}^6(\tilde{y})}{(|\tilde{x} - \tilde{y}|^2 + x_3^2)^3} dx_3 \right)^{1/6} d\tilde{y} \right]^6 d\tilde{x} \\ &= C \int \left[\int \frac{\tilde{\rho}(\tilde{y})}{|\tilde{x} - \tilde{y}|^{5/6}} d\tilde{y} \right]^6 d\tilde{x} = C \|\tilde{\rho} * |\cdot|^{-5/6}\|_{L^6(\mathbb{R}^2)}^6 \\ &\leq C \|\tilde{\rho}\|_{L^{4/3}(\mathbb{R}^2)}^6 \| |\cdot|^{-5/6} \|_{L_w^{12/5}(\mathbb{R}^2)}^6; \end{aligned}$$

the function $|\cdot|^{-\lambda}$ is an element of the weak L^p space $L_w^{n/\lambda}(\mathbb{R}^n)$, cf. [20, 4.3]. \square

We also need to investigate the integrability of U_ρ , restricted to the (x_1, x_2) plane.

Lemma 3.3. *There exists a bounded linear operator*

$$S : L^{6/5}(\mathbb{R}^3) \rightarrow L^4(\mathbb{R}^2)$$

such that for $\rho \in C_c(\mathbb{R}^3)$ compactly supported and continuous, $S\rho = U_\rho(\cdot, 0)$; notice that for such ρ the induced potential U_ρ is a continuous, pointwise defined function.

We write $U_\rho(\cdot, 0) = S\rho \in L^4(\mathbb{R}^2)$ also for $\rho \in L^{6/5}(\mathbb{R}^3)$ so that

$$\|U(\cdot, 0)\|_{L^4(\mathbb{R}^2)} \leq C \|\rho\|_{L^{6/5}(\mathbb{R}^3)}.$$

If in addition $\tilde{\rho} \in L^{4/3}(\mathbb{R}^2)$, then the following mixed potential energies exist and are equal:

$$\int U_{\tilde{\rho}}(\tilde{x})\rho(x) dx = \int U_\rho(\tilde{x}, 0)\tilde{\rho}(\tilde{x}) d\tilde{x}.$$

Proof. Fubini’s theorem together with the Hölder inequality and Lemma 3.2 imply that for $\rho \in C_c(\mathbb{R}^3)$, $\tilde{\rho} \in C_c(\mathbb{R}^2)$,

$$\begin{aligned} \int |U_\rho(\tilde{x}, 0)\tilde{\rho}(\tilde{x})| d\tilde{x} &\leq \iint \frac{|\rho(y)\tilde{\rho}(\tilde{x})|}{|(\tilde{x}, 0) - y|} d\tilde{x} dy = \int |U_{|\tilde{\rho}|}(y)\rho(y)| dy \\ &\leq C \|\rho\|_{L^{6/5}(\mathbb{R}^3)} \|\tilde{\rho}\|_{L^{4/3}(\mathbb{R}^2)}. \end{aligned}$$

The estimate for $\|U_\rho(\cdot, 0)\|_{L^4(\mathbb{R}^2)}$ follows by taking the supremum over all $\tilde{\rho} \in C_c(\mathbb{R}^2)$ with $\|\tilde{\rho}\|_{L^{4/3}(\mathbb{R}^2)} = 1$. This shows that the operator S is bounded with respect to the indicated norms on the dense subset $C_c(\mathbb{R}^3)$ of $L^{6/5}(\mathbb{R}^3)$ so that it can be extended as stated. Since the mixed potential energies now exist they are equal again by Fubini’s theorem. \square

It will be useful to view the potential energy as a bilinear form which induces a scalar product. More precisely we define for $\rho, \sigma \in L^{6/5}(\mathbb{R}^3)$,

$$\langle \rho, \sigma \rangle_{\text{pot}} := \frac{1}{2} \int \frac{\rho(x)\sigma(y)}{|x - y|} dy dx$$

with the analogous definition for $\langle \tilde{\rho}, \tilde{\sigma} \rangle_{\text{pot}}$, $\tilde{\rho}, \tilde{\sigma} \in L^{4/3}(\mathbb{R}^2)$, and

$$\langle \rho, \tilde{\rho} \rangle_{\text{pot}} := \frac{1}{2} \int \frac{\rho(x)\tilde{\rho}(\tilde{y})}{|x - (\tilde{y}, 0)|} d\tilde{y} dx. \tag{3.1}$$

It is well known that $\langle \cdot, \cdot \rangle_{\text{pot}}$ is a scalar product on $L^{6/5}(\mathbb{R}^3)$, cf. [20, 9.8], and the same is true on $L^{4/3}(\mathbb{R}^2)$. The induced norms are denoted by

$$\|\rho\|_{\text{pot}} := \langle \rho, \rho \rangle_{\text{pot}}^{1/2}, \quad \|\tilde{\rho}\|_{\text{pot}} := \langle \tilde{\rho}, \tilde{\rho} \rangle_{\text{pot}}^{1/2}.$$

Finally, $\langle f, g \rangle_{\text{pot}} := \langle \rho_f, \rho_g \rangle_{\text{pot}}$ etc., provided that the induced spatial densities belong to the proper L^p space, so that with this notation,

$$E_{\text{pot}}(f) = -\langle f, f \rangle_{\text{pot}} = -\|f\|_{\text{pot}}^2 \tag{3.2}$$

etc. The Cauchy-Schwarz inequality corresponding to the mixed case (3.1) is established next. It tells us how strong the mixed potential energy term is in comparison to the potential energies of its individual components.

Lemma 3.4. *Let $\rho \in L_+^{6/5}(\mathbb{R}^3)$, $\tilde{\rho} \in L_+^{4/3}(\mathbb{R}^2)$. Then*

$$|\langle \rho, \tilde{\rho} \rangle_{\text{pot}}| \leq \|\rho\|_{\text{pot}} \|\tilde{\rho}\|_{\text{pot}}.$$

Proof. We first show the assertion under the additional assumption that $\rho, \tilde{\rho} \in C_c^\infty$ are compactly supported and smooth. In that case U_ρ is smooth and bounded. Let $d \in C_c^\infty(\mathbb{R})$ be such that $d \geq 0$ and $\int d = 1$, and let $\delta^\epsilon(x) := \epsilon^{-1}d(x/\epsilon)$ denote the induced δ -sequence; $\epsilon > 0$. Then

$$\lim_{\epsilon \rightarrow 0} \int U_\rho(\tilde{x}, x_3) \delta^\epsilon(x_3) dx_3 = U_\rho(\tilde{x}, 0)$$

pointwise for $\tilde{x} \in \mathbb{R}^2$, while the latter integral is bounded in modulus by $\|U_\rho\|_\infty$. Using Lebesgue’s theorem and the fact that $\langle \cdot, \cdot \rangle_{\text{pot}}$ is a scalar product on $L^{6/5}(\mathbb{R}^3)$ we can now argue as follows:

$$\begin{aligned} |\langle \rho, \tilde{\rho} \rangle_{\text{pot}}| &= \left| \frac{1}{2} \int U_\rho(\tilde{x}, 0) \tilde{\rho}(\tilde{x}) d\tilde{x} \right| = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left| \iint U_\rho(\tilde{x}, x_3) \delta^\epsilon(x_3) \tilde{\rho}(\tilde{x}) dx_3 d\tilde{x} \right| \\ &= \lim_{\epsilon \rightarrow 0} |\langle \rho, \tilde{\rho} \otimes \delta^\epsilon \rangle_{\text{pot}}| \\ &\leq \|\rho\|_{\text{pot}} \lim_{\epsilon \rightarrow 0} \left(\frac{1}{2} \iint \frac{\tilde{\rho}(\tilde{x}) \tilde{\rho}(\tilde{y}) \delta^\epsilon(x_3) \delta^\epsilon(y_3)}{|x - y|} dx dy \right)^{1/2} \\ &\leq \|\rho\|_{\text{pot}} \lim_{\epsilon \rightarrow 0} \left(\frac{1}{2} \iint \frac{\tilde{\rho}(\tilde{x}) \tilde{\rho}(\tilde{y}) \delta^\epsilon(x_3) \delta^\epsilon(y_3)}{|\tilde{x} - \tilde{y}|} dx dy \right)^{1/2} \\ &= \|\rho\|_{\text{pot}} \|\tilde{\rho}\|_{\text{pot}}; \end{aligned}$$

in the last step we used that $\int \delta^\epsilon = 1$ for $\epsilon > 0$. The general case follows by approximating ρ and $\tilde{\rho}$ by compactly supported, smooth functions, observing the fact that both sides of the inequality are continuous with respect to the $L^{6/5}(\mathbb{R}^3)$ -norm for ρ and the $L^{4/3}(\mathbb{R}^2)$ -norm for $\tilde{\rho}$. \square

4. The Decoupled Minimizers

In the next sections the existence and properties of the minimizers of the decoupled problems where one of the components is missing will become important. Here we briefly collect the relevant facts. A function g on $\mathbb{R}^d \times \mathbb{R}^d$ is called spherically symmetric if for every $A \in \text{SO}(d)$, $g(Ax, Av) = g(x, v)$.

For each $M, N > 0$ the energy $\mathcal{H}(f)$ has a minimizer f_0^{3D} in the set

$$\mathcal{F}_{M,N}^{3D} := \left\{ f \in L_+^1(\mathbb{R}^6) \mid \|f\|_1 \leq M, \|f\|_{1+1/k} \leq N, E_{\text{kin}}(f) < \infty \right\}.$$

The minimizer is unique up to spatial shifts, spherically symmetric, has negative energy, i.e., $\mathcal{H}(f_0^{3D}) < 0$, saturates the constraints, i.e., $\|f_0^{3D}\|_1 = M, \|f_0^{3D}\|_{1+1/k} = N$, and has compact spatial support. There exists a constant $R^* > 0$ which is independent of M and N such that the radius of this spatial support is

$$R = R^* M^{(2k-1)/3} N^{-(2k+2)/3}. \tag{4.1}$$

By spherical symmetry,

$$f_0^{3D}(\tilde{x}, x_3, \tilde{v}, v_3) = f_0^{3D}(\tilde{x}, -x_3, \tilde{v}, -v_3).$$

Similarly, the energy $\mathcal{H}(\tilde{f})$ has a minimizer f_0^{FL} in the set

$$\mathcal{F}_{M,N}^{\text{FL}} := \left\{ \tilde{f} \in L_+^1(\mathbb{R}^4) \mid \|\tilde{f}\|_1 \leq \tilde{M}, \|\tilde{f}\|_{1+1/\tilde{k}} \leq \tilde{N}, E_{\text{kin}}(\tilde{f}) < \infty \right\}.$$

A slight complication arises from the fact that we do at the moment not know whether this minimizer is again unique up to spatial shifts. However, there does exist a two-parameter family $(f_{M,N}^{\text{FL}})_{M,N>0}$ such that $f_{M,N}^{\text{FL}}$ is a minimizer of $\mathcal{H}(\tilde{f})$ over $\mathcal{F}_{M,N}^{\text{FL}}$ which saturates the constraints, has negative energy, is axially symmetric with respect to the x_3 axis, i.e., spherically symmetric as a function of \tilde{x}, \tilde{v} , and has compact spatial support. There exists a constant \tilde{R}^* independent of M and N such that the radius of this spatial support is

$$\tilde{R} = \tilde{R}^* M^{\tilde{k}} N^{-(\tilde{k}+1)}. \tag{4.2}$$

In what follows f_0^{FL} always denotes the corresponding member of the above family. In particular, if

$$\mathbf{M} = (M, N, \tilde{M}, \tilde{N}) =: (\mathbf{M}^{3D}, \mathbf{M}^{\text{FL}}),$$

then f_0^{3D} denotes the minimizer of \mathcal{H} over $\mathcal{F}_{M,N}^{3D}$ and f_0^{FL} denotes $f_{\tilde{M},\tilde{N}}^{\text{FL}}$.

Since the above facts are known or follow by arguments already available in the literature we defer their discussion to the Appendix.

5. Properties of \mathcal{H}

First we establish a lower bound for \mathcal{H} on $\mathcal{F}_{\mathbf{M}}$ and certain a-priori bounds along minimizing sequences.

Lemma 5.1. (a) *The functional \mathcal{H} is bounded from below on $\mathcal{F}_{\mathbf{M}}$, i.e.,*

$$-\infty < \inf_{\mathcal{F}_{\mathbf{M}}} \mathcal{H} =: h_{\mathbf{M}} < 0.$$

(b) *Along every minimizing sequence $(f_j, \tilde{f}_j) \subset \mathcal{F}_{\mathbf{M}}$ of \mathcal{H} both the kinetic and the potential energies are bounded, more precisely, for j sufficiently large,*

$$E_{\text{kin}}(f_j) + E_{\text{kin}}(\tilde{f}_j) + |E_{\text{pot}}(f_j)| + |E_{\text{pot}}(\tilde{f}_j)| \leq C_{\mathbf{M}},$$

where the constant $C_{\mathbf{M}} > 0$ depends only on \mathbf{M} .

Proof. Lemma 3.1 and Lemma 3.4 imply that for $(f, \tilde{f}) \in \mathcal{F}_{\mathbf{M}}$,

$$\begin{aligned} \left| \langle f, \tilde{f} \rangle_{\text{pot}} \right| &\leq \|f\|_{\text{pot}} \|\tilde{f}\|_{\text{pot}} \leq C_{\mathbf{M}} E_{\text{kin}}(f)^{1/4} E_{\text{kin}}(\tilde{f})^{1/4} \\ &\leq C_{\mathbf{M}} E_{\text{kin}}(f)^{1/2} + C_{\mathbf{M}} E_{\text{kin}}(\tilde{f})^{1/2}; \end{aligned}$$

the value of the constant $C_{\mathbf{M}}$ may change from line to line. Using Lemma 3.1 again this yields the estimate

$$\mathcal{H}(f, \tilde{f}) \geq E_{\text{kin}}(f) - C_{\mathbf{M}} E_{\text{kin}}(f)^{1/2} + E_{\text{kin}}(\tilde{f}) - C_{\mathbf{M}} E_{\text{kin}}(\tilde{f})^{1/2}. \tag{5.1}$$

Hence $h_{\mathbf{M}} > -\infty$. Moreover,

$$h_{\mathbf{M}} \leq \mathcal{H}(f_0^{3\text{D}}, f_0^{\text{FL}}) = \mathcal{H}(f_0^{3\text{D}}) + \mathcal{H}(f_0^{\text{FL}}) + \int \tilde{U}_0 \rho_0 dx < 0.$$

Hence along a minimizing sequence $\mathcal{H}(f_j, \tilde{f}_j) \leq 0$ for j sufficiently large, and by (5.1),

$$\left(E_{\text{kin}}(f_j)^{1/2} - C_{\mathbf{M}}/2 \right)^2 + \left(E_{\text{kin}}(\tilde{f}_j)^{1/2} - C_{\mathbf{M}}/2 \right)^2 \leq C_{\mathbf{M}}^2/2.$$

Another reference to Lemma 3.1 completes the proof. \square

In order to pass to the limit along a minimizing sequence we need the following compactness properties of the potential energies; by $\mathbf{1}_S$ we denote the indicator function of the set S , and we recall (3.2) and the corresponding notation.

Lemma 5.2. *Let $(\rho_j) \subset L^{1+1/n}(\mathbb{R}^3)$ and $(\tilde{\rho}_j) \subset L^{1+1/\tilde{n}}(\mathbb{R}^2)$ be such that*

$$\rho_j \rightharpoonup \rho_0 \text{ weakly in } L^{1+1/n}(\mathbb{R}^3), \quad \tilde{\rho}_j \rightharpoonup \tilde{\rho}_0 \text{ weakly in } L^{1+1/\tilde{n}}(\mathbb{R}^2).$$

Then for each $R > 0$,

$$\|\mathbf{1}_{B_R}(\rho_j - \rho_0)\|_{\text{pot}} \rightarrow 0, \quad \|\mathbf{1}_{\tilde{B}_R}(\tilde{\rho}_j - \tilde{\rho}_0)\|_{\text{pot}} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Proof. The convergence of the non-flat potential energy is proved for example in [27, Lemma 2.5]. For the flat case we refer to [10, Lemma 3.6]. \square

A crucial step in the analysis is to show that minimizing sequences do not spread out in space and that up to spatial shifts not all the mass can leak out to infinity. This is the content of the next result.

Proposition 5.3. *Let $(f_j, \tilde{f}_j) \subset \mathcal{F}_M$ be a minimizing sequence of \mathcal{H} . Then there exists a sequence $(\tilde{a}_j) \subset \mathbb{R}^2$ of shift vectors, $\epsilon_0 > 0$, and $R_0 > 0$ such that for all sufficiently large $j \in \mathbb{N}$,*

$$\int_{(\tilde{a}_j, 0) + B_{R_0}} \int f_j \, dv \, dx \geq \epsilon_0, \quad \int_{\tilde{a}_j + \tilde{B}_{R_0}} \int \tilde{f}_j \, d\tilde{v} \, d\tilde{x} \geq \epsilon_0.$$

Here B_{R_0} and \tilde{B}_{R_0} denote the closed ball of radius R_0 about the origin in \mathbb{R}^3 or \mathbb{R}^2 respectively.

Remark. It is important that the same shift vectors work for both the non-flat and the flat component.

Proof. Let $U_j := U_{f_j}$, $\tilde{\rho}_j := \rho_{\tilde{f}_j}$, and let R^{3D} and R^{FL} denote the radii of the decoupled minimizers f_0^{3D} and f_0^{FL} subject to constraints \mathbf{M}^{3D} and \mathbf{M}^{FL} , cf. Sect. 4. Since

$$\lim_{j \rightarrow \infty} \mathcal{H}(f_j, \tilde{f}_j) \leq \mathcal{H}(f_0^{3D}, f_0^{FL}) = \mathcal{H}(f_0^{3D}) + \mathcal{H}(f_0^{FL}) + \int U_0^{3D} \rho_0^{FL} \, d\tilde{x},$$

we get that for j sufficiently large,

$$\mathcal{H}(f_j) + \mathcal{H}(\tilde{f}_j) + \int U_j \tilde{\rho}_j \, d\tilde{x} < \mathcal{H}(f_0^{3D}) + \mathcal{H}(f_0^{FL}) + \frac{1}{2} \int U_0^{3D} \rho_0^{FL} \, d\tilde{x}.$$

Since $\mathcal{H}(f_j) \geq \mathcal{H}(f_0^{3D})$ and $\mathcal{H}(\tilde{f}_j) \geq \mathcal{H}(f_0^{FL})$ this implies that

$$\begin{aligned} \int U_j \tilde{\rho}_j \, d\tilde{x} &\leq \frac{1}{2} \int U_0^{3D} \rho_0^{FL} \, d\tilde{x} = -\frac{1}{2} \iint \frac{\rho_0^{3D}(y) \rho_0^{FL}(\tilde{x})}{|(\tilde{x}, 0) - y|} \, d\tilde{x} \, dy \\ &< -\frac{M\tilde{M}}{2(R^{3D} + R^{FL})} \end{aligned} \tag{5.2}$$

for all sufficiently large $j \in \mathbb{N}$.

For $R > 1$ we write

$$\begin{aligned} \frac{1}{|x|} &= \mathbf{1}_{\{|x| \leq 1/R\}}(x) \frac{1}{|x|} + \mathbf{1}_{\{1/R < |x| < R\}}(x) \frac{1}{|x|} + \mathbf{1}_{\{|x| \geq R\}}(x) \frac{1}{|x|} \\ &=: K_R^1(x) + K_R^2(x) + K_R^3(x). \end{aligned}$$

With this splitting

$$\left| \int U_j \tilde{\rho}_j \, d\tilde{x} \right| = \iint \frac{\rho_j(y) \tilde{\rho}_j(\tilde{x})}{|(\tilde{x}, 0) - y|} \, d\tilde{x} \, dy = J_1 + J_2 + J_3.$$

The second and third terms are estimated straightforwardly:

$$\begin{aligned} J_2 &\leq R \iint_{|(\tilde{x}, 0) - y| < R} \rho_j(y) \tilde{\rho}_j(\tilde{x}) \, d\tilde{x} \, dy, \\ J_3 &\leq R^{-1} \iint \rho_j(y) \tilde{\rho}_j(\tilde{x}) \, d\tilde{x} \, dy \leq M\tilde{M}R^{-1}. \end{aligned}$$

For J_1 we first apply the Hölder inequality and then the general form of the Minkowski inequality as in the proof of Lemma 3.2 to obtain the estimate

$$\begin{aligned} J_1 &\leq \|\rho_j\|_{1+1/n} \left\| \int_{|\tilde{x}-\cdot|<1/R} \frac{\tilde{\rho}_j(\tilde{x})}{|(\tilde{x}, 0) - \cdot|} d\tilde{x} \right\|_{n+1} \\ &\leq C \|\rho_j\|_{1+1/n} \|\tilde{\rho}_j * (\tilde{K}_R^1)^{n/(n+1)}\|_{n+1} \\ &\leq C \|\rho_j\|_{1+1/n} \|\tilde{\rho}_j\|_{1+1/\tilde{n}} \|(\tilde{K}_R^1)^{n/(n+1)}\|_\gamma \\ &\leq C \|\rho_j\|_{1+1/n} \|\tilde{\rho}_j\|_{1+1/\tilde{n}} R^{-\sigma} \leq C_M R^{-\sigma}, \end{aligned}$$

where

$$\gamma := \left(\frac{1}{n+1} + \frac{1}{\tilde{n}+1} \right)^{-1} > 1, \quad \sigma := \frac{2}{\gamma} - \frac{n}{n+1} > 0;$$

recall that $3/2 < n < 5$ and $1 < \tilde{n} < 3$. With (5.2) we find that

$$-\frac{M\tilde{M}}{2(R^{3D} + R^{FL})} > \int U_j \tilde{\rho}_j d\tilde{x} = -J_1 - J_2 - J_3,$$

and hence

$$J_2 \geq \frac{M\tilde{M}}{2(R^{3D} + R^{FL})} - \frac{M\tilde{M}}{R} - C_M R^{-\sigma}.$$

For R sufficiently large the right hand side is positive, so that

$$0 < R^{-1} \left(\frac{M\tilde{M}}{2(R^{3D} + R^{FL})} - \frac{M\tilde{M}}{R} - C_M R^{-\sigma} \right) \leq \iint_{|x-(\tilde{y},0)|<R} \rho_j(x) \tilde{\rho}_j(\tilde{y}) dx d\tilde{y}. \tag{5.3}$$

The existence of the shift vectors (\tilde{a}_j) with the asserted properties is now a consequence of the following lemma. \square

Lemma 5.4. *Let $\rho \in L^1_+(\mathbb{R}^3)$, $\sigma \in L^1_+(\mathbb{R}^2)$ with*

$$0 < \int_{\mathbb{R}^3} \rho(x) dx, \int_{\mathbb{R}^2} \sigma(\tilde{y}) d\tilde{y} \leq M < \infty$$

and such that for some $\delta_0, r_0 > 0$,

$$\iint_{|x-(\tilde{y},0)|<r_0} \rho(x) \sigma(\tilde{y}) dx d\tilde{y} > \delta_0.$$

Then there exist $\epsilon_0, R_0 > 0$ depending only on δ_0, r_0 , and M such that

$$\epsilon_0 < \int_{|x-(\tilde{a},0)|<R_0} \rho(x) dx \text{ and } \epsilon_0 < \int_{|\tilde{y}-\tilde{a}|<R_0} \sigma(\tilde{y}) d\tilde{y}$$

for some $\tilde{a} \in \mathbb{R}^2$.

Proof. Let $z \in \mathbb{R}^3$ be given. Note first that

$$\begin{aligned} & \left\{ (x, \tilde{y}) \in \mathbb{R}^5 \mid |z - (\tilde{y}, 0)| < r_0, |x - (\tilde{y}, 0)| < r_0 \right\} \\ & \subset \left\{ (x, \tilde{y}) \in \mathbb{R}^5 \mid |z - x| < 2r_0, |x - (\tilde{y}, 0)| < r_0 \right\}, \end{aligned}$$

and hence

$$\begin{aligned} & \int_{|z - (\tilde{y}, 0)| < r_0} \left(\int_{|x - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) dx \right) d\tilde{y} \\ & \leq \int_{|z - x| < 2r_0} \left(\int_{|x - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) d\tilde{y} \right) dx. \end{aligned}$$

Multiplying with $\rho(z)$ and integrating with respect to $z \in \mathbb{R}^3$ we obtain

$$\begin{aligned} & \int \rho(z) \int_{|z - (\tilde{y}, 0)| < r_0} \left(\int_{|x - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) dx \right) d\tilde{y} dz \\ & \leq \int \rho(z) \int_{|z - x| < 2r_0} \left(\int_{|x - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) d\tilde{y} \right) dx dz. \end{aligned} \quad (5.4)$$

Changing the order of integration, the right hand side of (5.4) can be rewritten as

$$\begin{aligned} & \int_{x \in \mathbb{R}^3} \int_{|z - x| < 2r_0} \rho(z) \left(\int_{|x - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) d\tilde{y} \right) dz dx \\ & = \int_{x \in \mathbb{R}^3} \left(\int_{|z - x| < 2r_0} \rho(z) dz \right) \left(\int_{|x - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) d\tilde{y} \right) dx \\ & = \int_{\mathbb{R}^3} R(x) S(x) dx, \end{aligned}$$

where the functions R and S are defined by

$$R(x) := \int_{|x - z| < 2r_0} \rho(z) dz, \quad S(x) := \int_{|x - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) d\tilde{y}. \quad (5.5)$$

From our hypothesis,

$$\begin{aligned} \delta_0 & < \int_{z \in \mathbb{R}^3} \rho(z) \int_{|z - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) d\tilde{y} dz \\ & = \frac{3}{4\pi r_0^3} \int_{z \in \mathbb{R}^3} \rho(z) \int_{|z - (\tilde{y}, 0)| < r_0} \int_{|x - (\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) dx d\tilde{y} dz, \end{aligned}$$

so that combining with (5.4) we are led to

$$\delta_0 < \frac{3}{4\pi r_0^3} \int_{\mathbb{R}^3} R(x) S(x) dx. \quad (5.6)$$

As a direct consequence of our definitions (5.5) we find that

$$\|R\|_\infty \leq M, \quad \|S\|_\infty \leq M. \quad (5.7)$$

Furthermore,

$$\begin{aligned}\|R\|_1 &= \int_{x \in \mathbb{R}^3} \int_{|z-x| < 2r_0} \rho(z) dz dx = \int_{z \in \mathbb{R}^3} \int_{|z-x| < 2r_0} \rho(z) dx dz \\ &\leq 8M \frac{4\pi}{3} r_0^3,\end{aligned}$$

and

$$\begin{aligned}\|S\|_1 &= \int_{x \in \mathbb{R}^3} \int_{|x-(\tilde{y},0)| < r_0} \sigma(\tilde{y}) d\tilde{y} dx = \int_{\tilde{y} \in \mathbb{R}^3} \int_{|x-(\tilde{y},0)| < r_0} \sigma(\tilde{y}) dx d\tilde{y} \\ &\leq M \frac{4\pi}{3} r_0^3.\end{aligned}$$

We may thus continue with (5.6) as follows:

$$\begin{aligned}\delta_0 &< \frac{3}{4\pi r_0^3} \int_{\mathbb{R}^3} R(x) S(x) dx \\ &\leq \frac{3}{4\pi r_0^3} \|RS\|_\infty \int_{\mathbb{R}^3} (R(x) S(x))^{1/2} dx \\ &\leq \frac{3}{4\pi r_0^3} \|RS\|_\infty^{1/2} \left(\int_{\mathbb{R}^3} R(x) dx \right)^{1/2} \left(\int_{\mathbb{R}^3} S(x) dx \right)^{1/2} \\ &\leq 2\sqrt{2}M \|RS\|_\infty^{1/2}.\end{aligned}$$

So there exists $a \in \mathbb{R}^3$ such that

$$R(a) S(a) > \left(\frac{\delta_0}{2\sqrt{2}M} \right)^2.$$

In view of (5.7) this implies that

$$R(a) > \frac{\delta_0^2}{8M^3} \quad \text{and} \quad S(a) > \frac{\delta_0^2}{8M^3}.$$

Finally we write $a = (\tilde{a}, a_3)$ with $\tilde{a} \in \mathbb{R}^2$, $a_3 \in \mathbb{R}$ and observe that

$$\int_{|\tilde{a}-\tilde{y}| < r_0} \sigma(\tilde{y}) d\tilde{y} \geq \int_{|(\tilde{a}, a_3)-(\tilde{y}, 0)| < r_0} \sigma(\tilde{y}) d\tilde{y} = S(a) > \frac{\delta_0^2}{8M^3}.$$

In addition, $S(a) > 0$ clearly implies $|a_3| < r_0$, so that

$$\int_{|z-(\tilde{a}, 0)| < 3r_0} \rho(z) dz \geq \int_{|a-z| < 2r_0} \rho(z) dz = R(a) > \frac{\delta_0^2}{8M^3},$$

which is exactly our claim with

$$\epsilon_0 := \frac{\delta_0^2}{8M^3}, \quad R_0 := 3r_0. \tag{5.8}$$

□

In what follows it is important to control the parameters ϵ_0 and R_0 in Proposition 5.3 if the constraint vector \mathbf{M} varies. This is the content of the following corollary.

Corollary 5.5. *Let the constraint vector \mathbf{M} satisfy the bounds*

$$0 < M_l \leq M \leq M_u, \quad 0 < \tilde{M}_l \leq \tilde{M} \leq \tilde{M}_u, \\ 0 < N_l \leq N \leq N_u, \quad 0 < \tilde{N}_l \leq \tilde{N} \leq \tilde{N}_u.$$

Then the parameters ϵ_0 and R_0 in Proposition 5.3 can be chosen independently of \mathbf{M} and (f_j, \tilde{f}_j) , depending only on the bounds \mathbf{M}_l and \mathbf{M}_u .

Proof. Under the given bounds on \mathbf{M} we can choose $R > 0$ depending only on these bounds such that the left hand side in (5.3) is bounded from below by a parameter $\delta_0 > 0$ also depending only on these bounds. To this end, observe that M and \tilde{M} are bounded both from below and above, $C_{\mathbf{M}}$ is bounded from above, and the radii R^{3D} and R^{FL} are bounded from above in view of (4.1) and (4.2). Given (5.8) this completes the proof. \square

The last tool needed for the proof of Theorem 2.1 is the fact that the energy infimum $h_{\mathbf{M}}$ is sub-additive in \mathbf{M} . While up to now all components of the constraint vector \mathbf{M} were strictly positive, this sub-additivity is for technical reasons needed also in situations where the flat or the non-flat component of a constraint vector vanishes, i.e., $\mathbf{M} = (\mathbf{M}^{3D}, \mathbf{M}^{FL})$ and $\mathbf{M}^{FL} = 0$ or $\mathbf{M}^{3D} = 0$. In such a case $h_{\mathbf{M}}$ is obviously taken to denote $\mathcal{H}(f_0^{3D})$ or $\mathcal{H}(f_0^{FL})$ respectively, where f_0^{3D} is the minimizer of \mathcal{H} over $\mathcal{F}_{\mathbf{M}^{3D}}^{3D}$ and f_0^{FL} is the one over $\mathcal{F}_{\mathbf{M}^{FL}}^{FL}$, cf. Sect. 4. We say that *the constraint vector $\mathbf{M} \in [0, \infty]^4$ is nontrivial*, if

$$(M > 0 \text{ and } N > 0) \text{ or } (\tilde{M} > 0 \text{ and } \tilde{N} > 0).$$

Proposition 5.6. *For all $\mathbf{M}_1, \mathbf{M}_2 \in [0, \infty]^4$,*

$$h_{\mathbf{M}_1 + \mathbf{M}_2} \leq h_{\mathbf{M}_1} + h_{\mathbf{M}_2}.$$

If both \mathbf{M}_1 and \mathbf{M}_2 are nontrivial, then this inequality is strict. If \mathbf{M}_1 satisfies uniform bounds from above and below as in Corollary 5.5, and if either this is true also for \mathbf{M}_2 or one component of \mathbf{M}_2 vanishes and the other one satisfies such uniform bounds, then there exists $\epsilon > 0$ depending only on these bounds such that

$$h_{\mathbf{M}_1 + \mathbf{M}_2} \leq h_{\mathbf{M}_1} + h_{\mathbf{M}_2} - \epsilon.$$

Proof. Consider two minimizing sequences $(f_j^1, \tilde{f}_j^1) \subset \mathcal{F}_{\mathbf{M}_1}$ and $(f_j^2, \tilde{f}_j^2) \subset \mathcal{F}_{\mathbf{M}_2}$ with

$$\mathcal{H}(f_j^1, \tilde{f}_j^1) \rightarrow h_{\mathbf{M}_1}, \quad \mathcal{H}(f_j^2, \tilde{f}_j^2) \rightarrow h_{\mathbf{M}_2}.$$

We can assume that the minimizing sequences are shifted in such a way that the assertions of Proposition 5.3 hold with $\epsilon_0^1, \epsilon_0^2, R_0^1$, and R_0^2 , and without spatial shifts. If one component of \mathbf{M}_1 or \mathbf{M}_2 vanishes, say the flat one of \mathbf{M}_2 , the corresponding trivial minimizing sequence $(f_0^{3D}, 0)$ need of course not be shifted, and we take for R_0^2 the radius of the minimizer f_0^{3D} and for ϵ_0^2 its mass.

By the Minkowski inequality, $(f_j^1 + f_j^2, \tilde{f}_j^1 + \tilde{f}_j^2) \in \mathcal{F}_{\mathbf{M}_1 + \mathbf{M}_2}$, and hence

$$\begin{aligned} h_{\mathbf{M}_1 + \mathbf{M}_2} &\leq \mathcal{H}(f_j^1 + f_j^2, \tilde{f}_j^1 + \tilde{f}_j^2) \\ &= \mathcal{H}(f_j^1, \tilde{f}_j^1) + \mathcal{H}(f_j^2, \tilde{f}_j^2) \\ &\quad - 2\langle f_j^1, f_j^2 \rangle_{\text{pot}} - 2\langle \tilde{f}_j^1, \tilde{f}_j^2 \rangle_{\text{pot}} - 2\langle f_j^1, \tilde{f}_j^2 \rangle_{\text{pot}} - 2\langle f_j^2, \tilde{f}_j^1 \rangle_{\text{pot}} \\ &\leq \mathcal{H}(f_j^1, \tilde{f}_j^1) + \mathcal{H}(f_j^2, \tilde{f}_j^2) \rightarrow h_{\mathbf{M}_1} + h_{\mathbf{M}_2}. \end{aligned}$$

If \mathbf{M}_1 and \mathbf{M}_2 have both at least one nontrivial component, say, in both cases the non-flat one, then the corresponding potential interaction energy is bounded away from zero so that the estimate above is strict:

$$\begin{aligned} \langle f_j^1, f_j^2 \rangle_{\text{pot}} &= \langle \rho_j^1, \rho_j^2 \rangle_{\text{pot}} \geq \frac{1}{2} \iint_{B_{R_0^1} \times B_{R_0^2}} \frac{\rho_j^1(x) \rho_j^2(y)}{|x - y|} dx dy \\ &\geq \frac{1}{2} \frac{\epsilon_0^1 \epsilon_0^2}{R_0^1 + R_0^2}. \end{aligned}$$

Assume now that we have positive uniform lower and upper bounds for \mathbf{M}_1 and \mathbf{M}_2 . Then the above lower bound for the interaction term is bounded from below by some $\epsilon > 0$ depending only on the uniform bounds on the constraint vectors, where we use Corollary 5.5 and in addition (4.1) if the flat component of \mathbf{M}_2 vanishes. If the non-flat component of \mathbf{M}_2 vanishes we use (4.2) instead. \square

Remark. The uniform sub-additivity is also valid if both \mathbf{M}_1 and \mathbf{M}_2 have exactly one nontrivial component which is uniformly bounded from below and above, but this case is not needed in what follows.

6. Proof of Theorem 2.1

Let $(f_j, \tilde{f}_j) \in \mathcal{F}_{\mathbf{M}}$ be a minimizing sequence for \mathcal{H} . We choose shift vectors $\tilde{a}_j \in \mathbb{R}^2$ such that the assertion of Proposition 5.3 holds. To keep the notation simple we redefine (f_j, \tilde{f}_j) as the minimizing sequence shifted by these vectors as in the statement of Theorem 2.1. Hence according to Proposition 5.3,

$$\epsilon_0 \leq \int_{|x| \leq R_0} \int f_j dv dx \leq M, \quad \epsilon_0 \leq \int_{|\tilde{x}| \leq R_0} \int \tilde{f}_j d\tilde{v} d\tilde{x} \leq \tilde{M}. \tag{6.1}$$

This new sequence is of course minimizing as well. The definition of $\mathcal{F}_{\mathbf{M}}$ implies the a-priori bounds

$$\|f_j\|_{1+1/k} \leq N, \quad \|\tilde{f}_j\|_{1+1/\tilde{k}} \leq \tilde{N}.$$

Hence after extracting a subsequence which we denote by the same symbol,

$$f_j \rightharpoonup f_0 \text{ weakly in } L^{1+1/k}(\mathbb{R}^6), \quad \tilde{f}_j \rightharpoonup \tilde{f}_0 \text{ weakly in } L^{1+1/\tilde{k}}(\mathbb{R}^4).$$

From this weak convergence it follows that

$$\|f_0\|_1 \leq M, \quad \|\tilde{f}_0\|_1 \leq \tilde{M}, \quad \|f_0\|_{1+1/k} \leq N, \quad \|\tilde{f}_0\|_{1+1/\tilde{k}} \leq \tilde{N},$$

and

$$E_{\text{kin}}(f_0) \leq \limsup_{j \rightarrow \infty} E_{\text{kin}}(f_j) < \infty, \quad E_{\text{kin}}(\tilde{f}_0) \leq \limsup_{j \rightarrow \infty} E_{\text{kin}}(\tilde{f}_j) < \infty.$$

By Lemma 3.1 the corresponding spatial densities $\rho_j := \rho_{f_j}$ and $\tilde{\rho}_j := \rho_{\tilde{f}_j}$ are bounded in $L^{1+1/n}(\mathbb{R}^3)$ or $L^{1+1/\tilde{n}}(\mathbb{R}^2)$ respectively. After extracting a subsequence again,

$$\rho_j \rightharpoonup \rho_0 \text{ weakly in } L^{1+1/n}(\mathbb{R}^3), \quad \tilde{\rho}_j \rightharpoonup \tilde{\rho}_0 \text{ weakly in } L^{1+1/\tilde{n}}(\mathbb{R}^2).$$

It is easy to see that in fact $\rho_0 = \rho_{f_0}$ and $\tilde{\rho}_0 = \rho_{\tilde{f}_0}$. The essential step is to prove that up to extracting yet another subsequence the potential energy terms converge, i.e.,

$$\|f_j - f_0\|_{\text{pot}} + \|\tilde{f}_j - \tilde{f}_0\|_{\text{pot}} \rightarrow 0 \text{ as } j \rightarrow \infty;$$

by Lemma 3.4 it then follows that also $\langle f_j - f_0, \tilde{f}_j - \tilde{f}_0 \rangle_{\text{pot}} \rightarrow 0$.

For $R > R_1 \geq R_0$ we define $B_{R_1, R} := \{x \in \mathbb{R}^3 \mid R_1 \leq |x| < R\}$ with the obvious definition of $\tilde{B}_{R_1, R}$, and we split the functions f_j and \tilde{f}_j as follows:

$$\begin{aligned} f_j &= \mathbf{1}_{B_{R_1} \times \mathbb{R}^3} f_j + \mathbf{1}_{B_{R_1, R} \times \mathbb{R}^3} f_j + \mathbf{1}_{B_{R, \infty} \times \mathbb{R}^3} f_j =: f_j^1 + f_j^2 + f_j^3, \\ \tilde{f}_j &= \mathbf{1}_{\tilde{B}_{R_1} \times \mathbb{R}^2} \tilde{f}_j + \mathbf{1}_{\tilde{B}_{R_1, R} \times \mathbb{R}^2} \tilde{f}_j + \mathbf{1}_{\tilde{B}_{R, \infty} \times \mathbb{R}^2} \tilde{f}_j =: \tilde{f}_j^1 + \tilde{f}_j^2 + \tilde{f}_j^3. \end{aligned}$$

Lemma 5.2 implies that for $R > R_1 \geq R_0$ fixed,

$$\|f_j^1 + f_j^2 - f_0^1 - f_0^2\|_{\text{pot}} + \|\tilde{f}_j^1 + \tilde{f}_j^2 - \tilde{f}_0^1 - \tilde{f}_0^2\|_{\text{pot}} \rightarrow 0 \text{ as } j \rightarrow \infty. \tag{6.2}$$

So we only need to show that for any $\epsilon > 0$ and R sufficiently large,

$$\liminf_{j \rightarrow \infty} \left(\|f_j^3\|_{\text{pot}} + \|\tilde{f}_j^3\|_{\text{pot}} \right) < \epsilon. \tag{6.3}$$

Once this is established we use the triangle inequality for $\|\cdot\|_{\text{pot}}$ to conclude that

$$\|f_j - f_0\|_{\text{pot}} \leq \|f_j^1 + f_j^2 - f_0^1 - f_0^2\|_{\text{pot}} + \|f_j^3\|_{\text{pot}} + \|f_0^3\|_{\text{pot}}.$$

We can surely find $R > 1$ such that the right hand side is as small as we want for j sufficiently large. Hence for $j \rightarrow \infty$,

$$E_{\text{pot}}(f_j) \rightarrow E_{\text{pot}}(f_0)$$

and with the same argument,

$$E_{\text{pot}}(\tilde{f}_j) \rightarrow E_{\text{pot}}(\tilde{f}_0).$$

Finally by Lemma 3.4,

$$\int \tilde{U}_j \rho_j dx \rightarrow \int \tilde{U}_0 \rho_0 dx,$$

and all together this implies that

$$\mathcal{H}(f_0, \tilde{f}_0) \leq \lim_{j \rightarrow \infty} \mathcal{H}(f_j, \tilde{f}_j) = h_{\mathbf{M}}.$$

This is the desired minimizing property of (f_0, \tilde{f}_0) .

We prove (6.3) by contradiction, so assume that (6.3) is false, i.e.

$$\exists \epsilon_1 > 0 \forall R > 1 \exists j_0 \in \mathbb{N} \forall j \geq j_0 : \|f_j^3\|_{\text{pot}} + \|\tilde{f}_j^3\|_{\text{pot}} \geq \epsilon_1.$$

Then we can choose a subsequence such that without change of labeling it satisfies either

$$\forall R > 1 \exists j_0 \in \mathbb{N} \forall j \geq j_0 : \|f_j^3\|_{\text{pot}} \geq \epsilon_1/2 \quad (6.4)$$

or

$$\forall R > 1 \exists j_0 \in \mathbb{N} \forall j \geq j_0 : \|\tilde{f}_j^3\|_{\text{pot}} \geq \epsilon_1/2. \quad (6.5)$$

In the following we consider the first case; the second one can be treated analogously. The contradiction is arrived at by splitting f_j and \tilde{f}_j as above and then using the uniform sub-additivity from Proposition 5.6. Let us denote

$$f_j^0 := \mathbf{1}_{B_{R_0} \times \mathbb{R}^3} f_j, \quad \tilde{f}_j^0 := \mathbf{1}_{\tilde{B}_{R_0} \times \mathbb{R}^2} \tilde{f}_j.$$

Since the splitting parameters satisfy the relation $R > R_1 \geq R_0$, (6.1) implies that

$$\epsilon_0 \leq \|f_j^0\|_1 \leq \|f_j^1\|_1 \leq M, \quad \epsilon_0 \leq \|\tilde{f}_j^0\|_1 \leq \|\tilde{f}_j^1\|_1 \leq \tilde{M}. \quad (6.6)$$

We also need uniform lower bounds for the $L^{1+1/k}$ -norm and $L^{1+1/\tilde{k}}$ -norm. By Lemma 3.1,

$$\|f_j^0\|_1 = \|\rho_j^0\|_1 \leq C(R_0) \|\rho_j^0\|_{1+1/n} \leq C(R_0) \|f_j^0\|_{1+1/k}^{(k+1)/(n+1)},$$

with an analogous estimate for \tilde{f}_j . Hence with (6.1),

$$0 < C(\epsilon_0) \leq \|f_j^1\|_{1+1/k} \leq N, \quad 0 < C(\epsilon_0) \leq \|\tilde{f}_j^1\|_{1+1/\tilde{k}} \leq \tilde{N}. \quad (6.7)$$

From the assumption (6.4) we now derive such bounds also for f_j^3 . By Lemma 3.1 with $\theta \in]0, 1[$ an interpolation parameter and $\sigma := (1-\theta)(1+k)/(1+n)$,

$$\|f\|_{\text{pot}}^2 = |E_{\text{pot}}(f)| \leq C \|\rho\|_{6/5}^2 \leq C \|\rho\|_1^{2\theta} \|\rho\|_{1+1/n}^{2(1-\theta)} \leq C \|f\|_1^{2\theta} \|f\|_{1+1/k}^{2\sigma}. \quad (6.8)$$

With $f = f_j^3$ this implies that

$$0 < C(\epsilon_1) \leq \|f_j^3\|_1 \leq M, \quad 0 < C(\epsilon_1) \leq \|f_j^3\|_{1+1/k} \leq N. \quad (6.9)$$

To arrive at a contradiction we insert the splitting of f_j and \tilde{f}_j into the energy functional:

$$\begin{aligned} \mathcal{H}(f_j, \tilde{f}_j) &= \mathcal{H}(f_j^1, \tilde{f}_j^1) + \mathcal{H}(f_j^2, \tilde{f}_j^2) + \mathcal{H}(f_j^3, \tilde{f}_j^3) \\ &\quad - 2\langle f_j^2, f_j^1 + f_j^3 \rangle_{\text{pot}} - 2\langle f_j^1, f_j^3 \rangle_{\text{pot}} - 2\langle \tilde{f}_j^2, \tilde{f}_j^1 + \tilde{f}_j^3 \rangle_{\text{pot}} - 2\langle \tilde{f}_j^1, \tilde{f}_j^3 \rangle_{\text{pot}} \\ &\quad - 2\langle f_j^2, \tilde{f}_j^1 + \tilde{f}_j^3 \rangle_{\text{pot}} - 2\langle f_j^1, \tilde{f}_j^3 \rangle_{\text{pot}} - 2\langle f_j^1 + f_j^3, \tilde{f}_j^2 \rangle_{\text{pot}} - 2\langle f_j^3, \tilde{f}_j^1 \rangle_{\text{pot}} \\ &=: \mathcal{H}(f_j^1, \tilde{f}_j^1) + \mathcal{H}(f_j^2, \tilde{f}_j^2) + \mathcal{H}(f_j^3, \tilde{f}_j^3) \\ &\quad - I_1 - I_2 - \tilde{I}_1 - \tilde{I}_2 - J_1 - J_2 - \tilde{J}_1 - \tilde{J}_2. \end{aligned} \quad (6.10)$$

Using the Cauchy-Schwarz inequality for $\langle \cdot, \cdot \rangle_{\text{pot}}$, i.e., Lemma 3.4 for the mixed terms J_1 and \tilde{J}_1 , and the boundedness of potential energies along the minimizing sequence, cf. Lemma 5.1, we obtain the estimates

$$\begin{aligned} I_1 + J_1 &\leq C \|f_j^2\|_{\text{pot}} \leq C \left(\|f_j^2 - f_0^2\|_{\text{pot}} + \|f_0^2\|_{\text{pot}} \right), \\ \tilde{I}_1 + \tilde{J}_1 &\leq C \|\tilde{f}_j^2\|_{\text{pot}} \leq C \left(\|\tilde{f}_j^2 - \tilde{f}_0^2\|_{\text{pot}} + \|\tilde{f}_0^2\|_{\text{pot}} \right). \end{aligned}$$

For $R > 2R_1$ and $x \in B_{R_1}$, $y \in B_{R,\infty}$ we note that

$$\frac{1}{|x - y|} \leq \frac{1}{|y| - R_1} \leq \frac{1}{|y| - |y|/2} = \frac{2}{|y|},$$

and we combine this with the Hölder inequality to estimate I_2 , \tilde{I}_2 , J_2 , and \tilde{J}_2 as follows:

$$\begin{aligned} I_2 &\leq 2 \int_{B_{R_1}} \rho_j(x) dx \int_{B_{R,\infty}} |y|^{-1} \rho_j(y) dy \leq C \|\rho_j\|_{6/5}^2 \left(\frac{R_1}{R} \right)^{1/2}, \\ \tilde{I}_2 &\leq 2 \int_{\tilde{B}_{R_1}} \tilde{\rho}_j(\tilde{x}) d\tilde{x} \int_{\tilde{B}_{R,\infty}} |\tilde{y}|^{-1} \tilde{\rho}_j(\tilde{y}) d\tilde{y} \leq C \|\tilde{\rho}_j\|_{4/3}^2 \left(\frac{R_1}{R} \right)^{1/2}, \\ J_2 &\leq 2 \int_{B_{R_1}} \rho_j(x) dx \int_{\tilde{B}_{R,\infty}} |\tilde{y}|^{-1} \tilde{\rho}_j(\tilde{y}) d\tilde{y} \leq C \|\rho_j\|_{6/5} \|\tilde{\rho}_j\|_{4/3} \left(\frac{R_1}{R} \right)^{1/2}, \\ \tilde{J}_2 &\leq 2 \int_{\tilde{B}_{R_1}} \tilde{\rho}_j(\tilde{x}) d\tilde{x} \int_{B_{R,\infty}} |y|^{-1} \rho_j(y) dy \leq C \|\rho_j\|_{6/5} \|\tilde{\rho}_j\|_{4/3} \left(\frac{R_1}{R} \right)^{1/2}. \end{aligned}$$

We wish to apply the uniform sub-additivity from Proposition 5.6 to the constraint vectors induced by (f_j^1, \tilde{f}_j^1) and (f_j^3, \tilde{f}_j^3) . To this end, let

$$\begin{aligned} \mathbf{M}_j &:= (\|f_j\|_1, \|f_j\|_{1+1/k}, \|\tilde{f}_j\|_1, \|\tilde{f}_j\|_{1+1/\tilde{k}}), \\ \mathbf{M}_j^i &:= (\|f_j^i\|_1, \|f_j^i\|_{1+1/k}, \|\tilde{f}_j^i\|_1, \|\tilde{f}_j^i\|_{1+1/\tilde{k}}), \quad i = 1, 2, 3. \end{aligned}$$

From (6.6), (6.7), (6.9) we have the required uniform bounds for \mathbf{M}_j^1 and (M_j^3, N_j^3) . With respect to \tilde{f}_j^3 we now distinguish two cases. Either this function also satisfies such non-zero uniform bounds or it is negligible, more precisely:

Case 1. $\exists \epsilon_2 > 0 \forall R > 1 \exists j_0 \in \mathbb{N} \forall j \geq j_0 : |E_{\text{pot}}(\tilde{f}_j^3)| \geq \epsilon_2$. In this case the analogue of the potential energy estimate (6.8) for \tilde{f}_j^3 implies that

$$0 < C(\epsilon_2) \leq \|\tilde{f}_j^3\|_1 \leq \tilde{M}, \quad 0 < C(\epsilon_2) \leq \|\tilde{f}_j^3\|_{1+1/\tilde{k}} \leq \tilde{N}.$$

So we have obtained uniform positive bounds for each entry of the quantities \mathbf{M}_j^1 and \mathbf{M}_j^3 from above and below. By Proposition 5.6,

$$\mathcal{H}(f_j^1, \tilde{f}_j^1) + \mathcal{H}(f_j^2, \tilde{f}_j^2) + \mathcal{H}(f_j^3, \tilde{f}_j^3) \geq h_{\mathbf{M}_j^1} + h_{\mathbf{M}_j^2} + h_{\mathbf{M}_j^3} \geq h_{\mathbf{M}} + \epsilon$$

with $\epsilon > 0$ independent of the splitting parameters $R > 2R_1$ and of j . Recalling (6.10) we find that

$$\begin{aligned} h_{\mathbf{M}} - \mathcal{H}(f_j, \tilde{f}_j) + \epsilon &\leq I_1 + I_2 + \tilde{I}_1 + \tilde{I}_2 + J_1 + J_2 + \tilde{J}_1 + \tilde{J}_2 \\ &\leq C_1 \left[\|f_0^2\|_{\text{pot}} + \|\tilde{f}_0^2\|_{\text{pot}} + \|f_j^2 - f_0^2\|_{\text{pot}} + \|\tilde{f}_j^2 - \tilde{f}_0^2\|_{\text{pot}} + (R_1/R)^{1/2} \right]. \end{aligned}$$

We choose $R_1 \geq R_0$ such that

$$C_1 (\|f_0^2\|_{\text{pot}} + \|\tilde{f}_0^2\|_{\text{pot}}) < \epsilon/4.$$

Next we choose $R > 2R_1$ such that $C_1(R_1/R)^{1/2} \leq \epsilon/4$. For j large,

$$h_{\mathbf{M}} - \mathcal{H}(f_j, \tilde{f}_j) + \epsilon \leq \frac{1}{2}\epsilon + C_1 \left[\|f_j^2 - f_0^2\|_{\text{pot}} + \|\tilde{f}_j^2 - \tilde{f}_0^2\|_{\text{pot}} \right],$$

and by (6.2) this contradicts the fact that (f_j, \tilde{f}_j) is minimizing.

Case 2. $\forall \epsilon > 0 \exists R_\epsilon > 1 \forall j_0 \in \mathbb{N} \exists j \geq j_0 : \|\tilde{f}_j^3\|_{\text{pot}} < \epsilon$, provided $R \geq R_\epsilon$. In this case we neglect \tilde{f}_j^3 in the sub-additivity argument and recall that Proposition 5.6 yields $\epsilon_2 > 0$ only depending on the bounds for \mathbf{M}_j^1 , M_j^3 , and N_j^3 such that

$$\begin{aligned} h_{\mathbf{M}_j^1} + h_{\mathbf{M}_j^2} + h_{(M_j^3, N_j^3, 0, 0)} &\geq h_{\mathbf{M}_j^1 + (M_j^3, N_j^3, 0, 0)} + \epsilon_2 + h_{\mathbf{M}_j^2} \\ &\geq h_{\mathbf{M}_j^1 + (M_j^3, N_j^3, 0, 0) + \mathbf{M}_j^2} + \epsilon_2 \\ &\geq h_{\mathbf{M}_j^1 + (M_j^3, N_j^3, 0, 0) + \mathbf{M}_j^2} + \epsilon_2 + h_{(0, 0, \tilde{M}_j^3, \tilde{N}_j^3)} \geq h_{\mathbf{M}_j} + \epsilon_2. \end{aligned}$$

By the assumption of the present case we can choose a subsequence which we keep on denoting as before such that $\|\tilde{f}_j^3\|_{\text{pot}} < \epsilon$ for all $j \in \mathbb{N}$, where ϵ will be determined in terms of ϵ_2 below; if necessary we increase R so that $R \geq R_\epsilon$. By Lemma 3.4,

$$|\langle \tilde{f}_j^3, f_j^3 \rangle_{\text{pot}}| \leq C\epsilon.$$

Together with the assumption of the present case,

$$\mathcal{H}(f_j^3, \tilde{f}_j^3) = \mathcal{H}(f_j^3) + \mathcal{H}(\tilde{f}_j^3) - 2\langle \tilde{f}_j^3, f_j^3 \rangle_{\text{pot}} \geq \mathcal{H}(f_j^3) - \epsilon^2 - C\epsilon.$$

We choose $R > 2R_1 \geq 2R_0$ such that in (6.10), $I_1 + \dots + \tilde{J}_2 < \epsilon$. Hence by (6.10),

$$\begin{aligned} \mathcal{H}(f_j, \tilde{f}_j) &\geq \mathcal{H}(f_j^1, \tilde{f}_j^1) + \mathcal{H}(f_j^2, \tilde{f}_j^2) + \mathcal{H}(f_j^3, \tilde{f}_j^3) - \epsilon \\ &\geq \mathcal{H}(f_j^1, \tilde{f}_j^1) + \mathcal{H}(f_j^2, \tilde{f}_j^2) + \mathcal{H}(f_j^3) - \epsilon^2 - C\epsilon \\ &\geq h_{\mathbf{M}_j^1} + h_{\mathbf{M}_j^2} + h_{(M_j^3, N_j^3, 0, 0)} - \epsilon^2 - C\epsilon \\ &\geq h_{\mathbf{M}_j} + \epsilon_2 - \epsilon^2 - C\epsilon. \end{aligned}$$

If ϵ is chosen properly in terms of ϵ_2 ,

$$\mathcal{H}(f_j, \tilde{f}_j) \geq h_{\mathbf{M}} + \epsilon_2/2 \text{ as } j \rightarrow \infty.$$

This contradicts the minimizing property of (f_j, \tilde{f}_j) . If one considers the case (6.5) instead of (6.4), all the arguments remain the same with the roles of the flat and non-flat components interchanged. The proof of Theorem 2.1 is complete.

7. Properties of the Minimizer

First we exclude the possibility that for a minimizer $f_0 = 0$ or $\tilde{f}_0 = 0$. Indeed the next result shows that the constraints are to some extent saturated by any minimizer.

Proposition 7.1. *Let $(f_0, \tilde{f}_0) \in \mathcal{F}_M$ be a minimizer of \mathcal{H} over \mathcal{F}_M . Then*

$$\begin{aligned} \|f_0\|_1 &= M \vee \|\tilde{f}_0\|_1 = \tilde{M}, \\ \|f_0\|_{1+1/k} &= N, \quad \|\tilde{f}_0\|_{1+1/\tilde{k}} = \tilde{N}. \end{aligned}$$

Proof. We define for $a, b, c, d, e > 0$ a rescaled state (f_0^*, \tilde{f}_0^*) as

$$f_0^*(x, v) := af_0(bx, cv), \quad \tilde{f}_0^*(\tilde{x}, \tilde{v}) := d\tilde{f}_0(b\tilde{x}, e\tilde{v});$$

because of the mixed potential energy term x and \tilde{x} must be scaled in the same way. Then

$$\begin{aligned} E_{\text{kin}}(f_0^*) &= ab^{-3}c^{-5}E_{\text{kin}}(f_0), \quad E_{\text{kin}}(\tilde{f}_0^*) = db^{-2}e^{-4}E_{\text{kin}}(\tilde{f}_0), \\ E_{\text{pot}}(f_0^*) &= a^2b^{-5}c^{-6}E_{\text{pot}}(f_0), \quad E_{\text{pot}}(\tilde{f}_0^*) = d^2b^{-3}e^{-4}E_{\text{pot}}(\tilde{f}_0), \\ &\int \tilde{U}_0^* \rho_0^* dx = adb^{-4}c^{-3}e^{-2} \int \tilde{U}_0 \rho_0 dx. \end{aligned}$$

Assume that $\|f_0\|_{1+1/k} < N$. Then we choose

$$a = c^3, \quad b = d = e = 1.$$

For this choice of parameters $\tilde{f}_0^* = \tilde{f}_0$,

$$\|f_0^*\|_1 = \|f_0\|_1, \quad \|f_0^*\|_{1+1/k} = c^{3/(k+1)}\|f_0\|_{1+1/k},$$

and

$$\mathcal{H}(f_0^*, \tilde{f}_0^*) = c^{-2}E_{\text{kin}}(f_0) + E_{\text{pot}}(f_0) + \mathcal{H}(\tilde{f}_0) + \int \tilde{U}_0 \rho_0 dx.$$

We can choose $c > 1$ so that the rescaled state still lies in \mathcal{F}_M and has lower energy which is a contradiction. The analogous argument shows that $\|\tilde{f}_0\|_{1+1/\tilde{k}} = \tilde{N}$.

In order to prove that at least one of the two mass constraints is saturated we assume that $\|f_0\|_1 < M \wedge \|\tilde{f}_0\|_1 < \tilde{M}$, and we choose the scaling parameters

$$a = c^{-7}, \quad b = d = c^{-4}, \quad e = c.$$

For this choice,

$$\begin{aligned} \|f_0^*\|_1 &= c^2\|f_0\|_1, \quad \|\tilde{f}_0^*\|_1 = c^2\|\tilde{f}_0\|_1, \\ \|f_0^*\|_{1+1/k} &= c^{(2k-7)/(k+1)}\|f_0\|_{1+1/k}, \quad \|\tilde{f}_0^*\|_{1+1/\tilde{k}} = c^{(2\tilde{k}-4)/(\tilde{k}+1)}\|\tilde{f}_0\|_{1+1/\tilde{k}}, \end{aligned}$$

and

$$\mathcal{H}(f_0^*, \tilde{f}_0^*) = \mathcal{H}(f_0, \tilde{f}_0).$$

Since $0 < k < 7/2$ and $0 < \tilde{k} < 2$ we can choose $c > 1$ such that $(f_0^*, \tilde{f}_0^*) \in \mathcal{F}_{\mathbf{M}}$ and both $\mathbf{M}^* := (\|f_0^*\|_1, \|f_0^*\|_{1+1/k}, \|\tilde{f}_0^*\|_1, \|\tilde{f}_0^*\|_{1+1/\tilde{k}})$ and $\mathbf{M} - \mathbf{M}^*$ are non-trivial. The strict sub-additivity in Proposition 5.6 implies the desired contradiction:

$$h_{\mathbf{M}} < h_{\mathbf{M}^*} + h_{\mathbf{M} - \mathbf{M}^*} < \mathcal{H}(f_0^*, \tilde{f}_0^*) = \mathcal{H}(f_0, \tilde{f}_0) = h_{\mathbf{M}}.$$

□

The main result of this section is the fact that the minimizers are functions of the particle or local energy. We use the Lagrange multiplier method presented for example in [4, 14, 15, 27, 29].

Theorem 7.2. *Let (f_0, \tilde{f}_0) be a minimizer as obtained in Theorem 2.1 with induced potentials (U_0, \tilde{U}_0) . Then*

$$\begin{aligned} f_0(x, v) &= \left(\frac{E_0 - E(x, v)}{\lambda} \right)_+^k \text{ a.e.}, \\ \tilde{f}_0(\tilde{x}, \tilde{v}) &= \left(\frac{\tilde{E}_0 - E(\tilde{x}, 0, \tilde{v}, 0)}{\tilde{\lambda}} \right)_+^{\tilde{k}} \text{ a.e.}, \end{aligned}$$

where $E(x, v) := \frac{1}{2}|v|^2 + U_0(x) + \tilde{U}_0(x)$ and $(\cdot)_+$ denotes the positive part. The Lagrange multipliers are defined as

$$\begin{aligned} E_0 &:= \frac{1}{\|f_0\|_1} \left(\frac{2k+5}{3} E_{\text{kin}}(f_0) + 2E_{\text{pot}}(f_0) + \int U_0 \tilde{\rho}_0 dx \right), \\ \tilde{E}_0 &:= \frac{1}{\|\tilde{f}_0\|_1} \left((\tilde{k}+2) E_{\text{kin}}(\tilde{f}_0) + 2E_{\text{pot}}(\tilde{f}_0) + \int U_0 \tilde{\rho}_0 dx \right), \end{aligned}$$

and

$$\lambda := \frac{2(k+1)E_{\text{kin}}(f_0)}{3\|f_0\|_{1+1/k}^{1+1/k}}, \quad \tilde{\lambda} := \frac{(\tilde{k}+1)E_{\text{kin}}(\tilde{f}_0)}{\|\tilde{f}_0\|_{1+1/\tilde{k}}^{1+1/\tilde{k}}}.$$

Proof. Let (f_0, \tilde{f}_0) be a minimizer of \mathcal{H} with corresponding potentials (U_0, \tilde{U}_0) . For f such that $(f, \tilde{f}_0) \in \mathcal{F}_{\mathbf{M}}$ we define

$$\mathcal{G}(f) := \mathcal{H}(f, \tilde{f}_0).$$

Then

$$\begin{aligned} \mathcal{G}(f) - \mathcal{G}(f_0) &= E_{\text{kin}}(f) - E_{\text{kin}}(f_0) + E_{\text{pot}}(f) - E_{\text{pot}}(f_0) \\ &\quad + \int (\rho_f - \rho_0) \tilde{U}_0 dx. \end{aligned} \tag{7.1}$$

For each fixed $\epsilon > 0$ we define the set

$$S_\epsilon := \left\{ (x, y) \in \mathbb{R}^6 \mid \epsilon \leq f_0(x, v) \leq \epsilon^{-1} \right\}.$$

Let $\eta \in L^\infty(\mathbb{R}^6)$ be a real-valued function with compact support such that $\eta \geq 0$ a.e. for $(x, v) \in \mathbb{R}^6 \setminus \text{supp } f_0$ and $\text{supp } \eta \subset (\mathbb{R}^6 \setminus \text{supp } f_0) \cup S_\epsilon$. For $t \in [0, T]$ and $T = (\|\eta\|_1 + \|\eta\|_{1+1/k} + \|\eta\|_\infty)^{-1} \epsilon/2$ we define

$$f_t(x, v) := \alpha^3(t) \|f_0\|_1 \frac{f_0 + t\eta}{\|f_0 + t\eta\|_1}(x, \alpha(t)v),$$

where

$$\alpha(t) := \left(\frac{\|f_0\|_{1+1/k}}{\|f_0\|_1} \frac{\|f_0 + t\eta\|_1}{\|f_0 + t\eta\|_{1+1/k}} \right)^{(k+1)/3}.$$

For $t \in [0, T]$,

$$\|f_t\|_1 = \|f_0\|_1, \quad \|f_t\|_{1+1/k} = \|f_0\|_{1+1/k}$$

and $f_0 + t\eta \geq 0$ a.e. For ϵ small enough,

$$\begin{aligned} \frac{\|f_0\|_1}{2} &\leq \|f_0 + t\eta\|_1 \leq \|f_0\|_1 + \frac{\epsilon}{2}, \\ \frac{\|f_0\|_{1+1/k}}{2} &\leq \|f_0 + t\eta\|_{1+1/k} \leq \|f_0\|_{1+1/k} + \frac{\epsilon}{2}, \end{aligned}$$

which implies that α is a smooth function on $[0, T]$ and

$$\alpha'(t) = \frac{k+1}{3} \alpha(t) \left[\frac{\|\eta\|_1}{\|f_0 + t\eta\|_1} - \frac{\iint (f_0 + t\eta)^{1/k} \eta \, dx \, dv}{\|f_0 + t\eta\|_{1+1/k}^{1+1/k}} \right].$$

Moreover, α'' is bounded on $[0, T]$. From (7.1) we conclude that for $t \in [0, T]$,

$$\begin{aligned} \mathcal{G}(f_t) - \mathcal{G}(f_0) &= \left(\frac{\|f_0\|_1}{\alpha^2(t) \|f_0 + t\eta\|_1} - 1 \right) E_{\text{kin}}(f_0) + \frac{\|f_0\|_1 t}{\alpha^2(t) \|f_0 + t\eta\|_1} E_{\text{kin}}(\eta) \\ &+ \left(\frac{\|f_0\|_1^2}{\|f_0 + t\eta\|_1^2} - 1 \right) E_{\text{pot}}(f_0) + \frac{\|f_0\|_1^2 t}{\|f_0 + t\eta\|_1^2} \int \rho_\eta U_0 \, dx \\ &+ \frac{\|f_0\|_1^2 t^2}{\|f_0 + t\eta\|_1^2} E_{\text{pot}}(\eta) + \left(\frac{\|f_0\|_1}{\|f_0 + t\eta\|_1} - 1 \right) \int \rho_0 \tilde{U}_0 \, dx \\ &+ \frac{\|f_0\|_1 t}{\|f_0 + t\eta\|_1} \int \rho_\eta \tilde{U}_0 \, dx. \end{aligned} \tag{7.2}$$

By Taylor expansion at $t = 0$,

$$\begin{aligned} \frac{\|f_0\|_1}{\alpha^2(t)\|f_0+t\eta\|_1} - 1 &= -t \left[\frac{\|\eta\|_1}{\|f_0\|_1} + 2\frac{k+1}{3} \left(\frac{\|\eta\|_1}{\|f_0\|_1} - \frac{\iint f_0^{1/k} \eta dx dv}{\|f_0\|_{1+1/k}^{1+1/k}} \right) \right] + O(t^2), \\ \frac{\|f_0\|_1 t}{\alpha^2(t)\|f_0+t\eta\|_1} &= t + O(t^2), \\ \frac{\|f_0\|_1^2}{\|f_0+t\eta\|_1^2} - 1 &= -\frac{2\|\eta\|_1 t}{\|f_0\|_1} + O(t^2), \\ \frac{\|f_0\|_1^2 t}{\|f_0+t\eta\|_1^2} &= t + O(t^2), \\ \frac{\|f_0\|_1}{\|f_0+t\eta\|_1} - 1 &= -\frac{\|\eta\|_1 t}{\|f_0\|_1} + O(t^2), \\ \frac{\|f_0\|_1 t}{\|f_0+t\eta\|_1} &= t + O(t^2). \end{aligned}$$

If we substitute these expansions into (7.2), we find that

$$\mathcal{G}(f_t) - \mathcal{G}(f_0) = t \iint (E - E_0 + \lambda f_0^{1/k}) \eta dv dx + O(t^2)$$

with E_0 and λ as given in the theorem. Since $\mathcal{G}(f_t)$ attains its minimum at $t = 0$, the choice of η and $\epsilon \rightarrow 0$ imply that $E - E_0 \geq 0$ on $\mathbb{R}^6 \setminus \text{supp } f_0$ and

$$f_0 = \left(\frac{E_0 - E}{\lambda} \right)^k \text{ a.e. on } \text{supp } f_0.$$

If we repeat this argument with the roles of flat and non-flat states exchanged, i.e., for $\mathcal{G}(\tilde{f}) := \mathcal{H}(f_0, \tilde{f})$, we obtain the assertion for \tilde{f}_0 . \square

The previous theorem states that for a minimizer (f_0, \tilde{f}_0) both components are functions of the local or particle energy in the induced potential $U_{0,e} = U_0 + \tilde{U}_0$. Since the latter is time-independent, the particle energy is conserved along particle orbits, i.e., along the characteristics of the Vlasov equations (1.1) and (1.2) respectively. Hence f_0 and \tilde{f}_0 satisfy these equations at least formally, and we are justified to refer to (f_0, \tilde{f}_0) as a steady state of the system (1.1)–(1.4). We do not discuss the regularity of this steady state further. However, to conclude this section we want to address the question whether these states have spatially compact support.

Proposition 7.3. *Let (f_0, \tilde{f}_0) be a minimizer as obtained in Theorem 2.1 and assume that*

$$0 < k < 5/2 \text{ and } 0 < \tilde{k} < 1.$$

Then $U_0, \tilde{U}_0, \rho_0, \tilde{\rho}_0 \in L^\infty(\mathbb{R}^3)$ with

$$\lim_{|x| \rightarrow \infty} U_0(x) = 0, \quad \lim_{|x| \rightarrow \infty} \tilde{U}_0(x) = 0,$$

$E_0, \tilde{E}_0 < 0$, and ρ_0 and $\tilde{\rho}_0$ have compact support.

Proof. Consider a density $\tilde{\rho} \in L^1_+ \cap L^p(\mathbb{R}^2)$. Then $U_{\tilde{\rho}} \in L^\infty(\mathbb{R}^3)$ with $\lim_{|x| \rightarrow \infty} U_{\tilde{\rho}}(x) = 0$, provided $p > 2$. If ρ is defined on \mathbb{R}^3 then the same is true provided $p > 3/2$. We prove this assertion for the flat case. Here

$$\begin{aligned} -U_{\tilde{\rho}}(x) &= \int_{\mathbb{R}^2} \frac{\tilde{\rho}(\tilde{y})}{|x - (\tilde{y}, 0)|} d\tilde{y} \\ &= \int_{|x - (\tilde{y}, 0)| \leq R} \frac{\tilde{\rho}(\tilde{y})}{|x - (\tilde{y}, 0)|} d\tilde{y} + \int_{|x - (\tilde{y}, 0)| > R} \frac{\tilde{\rho}(\tilde{y})}{|x - (\tilde{y}, 0)|} d\tilde{y}. \\ &\leq \int_{|\tilde{x} - \tilde{y}| \leq \sqrt{R^2 - x_3^2}} \frac{\tilde{\rho}(\tilde{y})}{|x - (\tilde{y}, 0)|} d\tilde{y} + \frac{\|\tilde{\rho}\|_1}{R} \\ &\leq CR^{(p-2)/p} \|\tilde{\rho}\|_{L^p(\{|\tilde{x} - \tilde{y}| \leq \sqrt{R^2 - x_3^2}\})} + \frac{\|\tilde{\rho}\|_1}{R}. \end{aligned}$$

Since this holds for any $R > 0$ the assertion follows; notice that for $R > 0$ fixed the first term goes to zero for $|x| \rightarrow \infty$. By the weak Young inequality and Lemma 3.3, $U_0 \in L^6(\mathbb{R}^3)$ and $U_0(\cdot, 0) \in L^4(\mathbb{R}^2)$, and again by the weak Young inequality and Lemma 3.2, $\tilde{U}_0 \in L^4(\mathbb{R}^2)$ and $\tilde{U}_0 \in L^6(\mathbb{R}^3)$. Hence $U_{0,e} \in L^4(\mathbb{R}^2) \cap L^6(\mathbb{R}^3)$. If we integrate the relations between f_0, \tilde{f}_0 , and $U_{0,e}$ from Theorem 7.2 with respect to v or \tilde{v} respectively we obtain the relations

$$\rho = c(E_0 - U_{0,e})^n_+, \quad \tilde{\rho} = \tilde{c}(\tilde{E}_0 - U_{0,e}(\cdot, 0))^{\tilde{n}}_+, \tag{7.3}$$

where c and \tilde{c} depend on λ and k or $\tilde{\lambda}$ and \tilde{k} respectively. From the integrability assertions for the potential we conclude that the spatial densities have the required integrability provided $6/n > 3/2$, i.e., $n < 4$ which means $k < 5/2$, and $4/\tilde{n} > 2$ i.e., $\tilde{n} < 2$ which means $\tilde{k} < 1$.

It therefore remains to show that $E_0 < 0$ and $\tilde{E}_0 < 0$ as claimed; the assertion on the support of the densities then follows. Assume that $E_0 > 0$. Then for $|x|$ large, $\rho_0(x) > c(E_0/2)^n$ which contradicts its integrability, and the same argument works for $\tilde{\rho}_0$. Now assume that $E_0 = 0$. Then $\rho_0(x) = c(-U_{0,e}(x))^n$, and this again contradicts the integrability of ρ_0 since $-U_{0,e} \geq C/|x|$ for large $|x|$ and $C > 0$. We prove this for \tilde{U}_0 , the argument for U_0 being completely analogous. We choose $R > 0$ such that

$$\int_{|\tilde{y}| \leq R} \tilde{\rho}_0(\tilde{y}) d\tilde{y} =: m > 0.$$

Next we observe that for $|\tilde{y}| \leq R$ and $|x| \geq 2R$,

$$\left| \frac{1}{|x - (\tilde{y}, 0)|} - \frac{1}{|x|} \right| \leq \frac{R}{(|x| - R)^2}.$$

If we restrict the convolution integral defining \tilde{U}_0 to the set $\{|y| \leq R\}$ and expand the kernel as indicated the assertion on \tilde{U}_0 follows. The same argument works for U_0 so that indeed $-U_{0,e} \geq C/|x|$ as claimed. If $\tilde{E}_0 = 0$ then $\tilde{\rho}_0(\tilde{x}) = c(-U_{0,e}(\tilde{x}, 0))^{\tilde{n}}$ which contradicts the integrability of $\tilde{\rho}$. Notice that under the present assumptions on k and \tilde{k} it follows that $n < 3$ and $\tilde{n} < 2$. \square

8. Stability

In this section we show how the minimizing property of a minimizer $(f_0, \tilde{f}_0) \in \mathcal{F}_{\mathbf{M}}$ leads to a stability estimate. Given a second state $(f, \tilde{f}) \in \mathcal{F}_{\mathbf{M}}$ and denoting the effective potential of the minimizer by $U_{0,e}$, a simple computation shows that

$$\begin{aligned} \mathcal{H}(f, \tilde{f}) &= \mathcal{H}(f_0, \tilde{f}_0) + \iint \left(\frac{1}{2}|v|^2 + U_{0,e}(x) \right) (f - f_0)(x, v) dv dx \\ &\quad + \iint \left(\frac{1}{2}|\tilde{v}|^2 + U_{0,e}(\tilde{x}, 0) \right) (\tilde{f} - \tilde{f}_0)(\tilde{x}, \tilde{v}) d\tilde{v} d\tilde{x} \\ &\quad - \|f - f_0\|_{\text{pot}}^2 - \|\tilde{f} - \tilde{f}_0\|_{\text{pot}}^2 - 2\langle f - f_0, \tilde{f} - \tilde{f}_0 \rangle_{\text{pot}}. \end{aligned}$$

With $E(x, v) = \frac{1}{2}|v|^2 + U_{0,e}(x)$ and

$$\begin{aligned} d((f, \tilde{f}), (f_0, \tilde{f}_0)) &:= \iint E(x, v) (f - f_0)(x, v) dv dx \\ &\quad + \iint E(\tilde{x}, 0, \tilde{v}, 0) (\tilde{f} - \tilde{f}_0)(\tilde{x}, \tilde{v}) d\tilde{v} d\tilde{x}, \end{aligned} \quad (8.1)$$

we can rewrite this expansion as

$$\begin{aligned} \mathcal{H}(f, \tilde{f}) &= \mathcal{H}(f_0, \tilde{f}_0) + d((f, \tilde{f}), (f_0, \tilde{f}_0)) \\ &\quad - \|f - f_0\|_{\text{pot}}^2 - \|\tilde{f} - \tilde{f}_0\|_{\text{pot}}^2 - 2\langle f - f_0, \tilde{f} - \tilde{f}_0 \rangle_{\text{pot}}. \end{aligned} \quad (8.2)$$

We need to show that $d((f, \tilde{f}), (f_0, \tilde{f}_0)) \geq 0$ with equality only if $(f, \tilde{f}) = (f_0, \tilde{f}_0)$. To this end we restrict ourselves to states $(f, \tilde{f}) \in \mathcal{F}_{\mathbf{M}}$ such that

$$\int f = \int f_0, \quad \int f^{1+1/k} = \int f_0^{1+1/k}, \quad \int \tilde{f} = \int \tilde{f}_0, \quad \int \tilde{f}^{1+1/\tilde{k}} = \int \tilde{f}_0^{1+1/\tilde{k}}. \quad (8.3)$$

Remark. From a physics point of view a galaxy and its halo are typically perturbed by the gravitational field of some distant exterior object. In particular, the perturbation will result in a measure preserving redistribution of the particles in phase space, and will hence preserve the constraints in (8.3). On the other hand, the fact that the perturbations lie in $\mathcal{F}_{\mathbf{M}}$ means that the stars are only shifted within the galactic plane and not perpendicularly to it. This is certainly an unphysical restriction. To remove it is a non-trivial problem for future research.

Using (8.3) and the strict convexity of the function $[0, \infty[\ni \zeta \mapsto \zeta^p$ for $p > 1$ we find that

$$\begin{aligned} d((f, \tilde{f}), (f_0, \tilde{f}_0)) &= \iint (E - E_0)(f - f_0) + \frac{\lambda}{1+1/k} \iint (f^{1+1/k} - f_0^{1+1/k}) \\ &\quad + \iint (E - \tilde{E}_0)(\tilde{f} - \tilde{f}_0) + \frac{\tilde{\lambda}}{1+1/\tilde{k}} \iint (\tilde{f}^{1+1/\tilde{k}} - \tilde{f}_0^{1+1/\tilde{k}}) \\ &\geq \iint \left[(E - E_0) + \lambda f_0^{1/k} \right] (f - f_0) \\ &\quad + \iint \left[(E - \tilde{E}_0) + \tilde{\lambda} \tilde{f}_0^{1/\tilde{k}} \right] (\tilde{f} - \tilde{f}_0) \geq 0; \end{aligned}$$

Theorem 7.2 implies that the last expressions are non-negative, and the strict convexity implies that equality holds only if $(f, \tilde{f}) = (f_0, \tilde{f}_0)$.

In order to establish a stability result, we now wish to apply the above estimates to the time evolution $(f(t), \tilde{f}(t))$ of a perturbation of (f_0, \tilde{f}_0) . Clearly, we need to require that $(f(0), \tilde{f}(0)) \in \mathcal{F}_M$ satisfies the constraints (8.3). More importantly, in view of the fact that nothing is known on the initial value problem for the system (1.1)–(1.4), we have to assume that this system has solutions $t \mapsto (f(t), \tilde{f}(t))$ which preserve the total energy, the constraints (8.3), and $(f(t), \tilde{f}(t)) \in \mathcal{F}_M$. To keep the rest of the discussion simple we furthermore assume that the minimizer (f_0, \tilde{f}_0) is unique in \mathcal{F}_M up to spatial shifts. If the minimizer is up to spatial shifts only isolated with respect to the distance measurement used in the stability estimate below, the result remains unchanged. If the minimizers are not even isolated one can prove the stability of the whole set of minimizers; we refer to [26] for the corresponding modifications of the arguments.

Stability estimate. *Assume the minimizer (f_0, \tilde{f}_0) is unique in \mathcal{F}_M . Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for any solution $t \mapsto (f(t), \tilde{f}(t))$ of (1.1)–(1.4) satisfying the above assumptions the following is true: If*

$$d((f(0), \tilde{f}(0)), (f_0, \tilde{f}_0)) + \|f(0) - f_0\|_{\text{pot}} + \|\tilde{f}(0) - \tilde{f}_0\|_{\text{pot}} < \delta,$$

then

$$d((f(t), \tilde{f}(t)), (f_0, \tilde{f}_0)) + \|f(t) - f_0\|_{\text{pot}} + \|\tilde{f}(t) - \tilde{f}_0\|_{\text{pot}} < \epsilon$$

up to spatial shifts parallel to the (x_1, x_2) plane and as long as the solution exists.

We do not call this a theorem because it is not clear that sufficiently regular solutions to the initial value problem do indeed exist. Assuming the latter the proof is by contradiction. If the assertion were false, there exists $\epsilon > 0$, a sequence of solutions $(t \mapsto (f_j(t), \tilde{f}_j(t)))$ and a sequence of times (t_j) such that for all $j \in \mathbb{N}$,

$$d((f_j(0), \tilde{f}_j(0)), (f_0, \tilde{f}_0)) + \|f_j(0) - f_0\|_{\text{pot}} + \|\tilde{f}_j(0) - \tilde{f}_0\|_{\text{pot}} < 1/j, \tag{8.4}$$

but

$$d((f_j(t_j), \tilde{f}_j(t_j)), (f_0, \tilde{f}_0)) + \|f_j(t_j) - f_0\|_{\text{pot}} + \|\tilde{f}_j(t_j) - \tilde{f}_0\|_{\text{pot}} \geq \epsilon, \tag{8.5}$$

regardless of how we shift $(f_j(t_j), \tilde{f}_j(t_j))$ in space. Now (8.4) and the fact that d is non-negative imply that all three terms in (8.4) converge to zero. By Lemma 3.4 this is true also for the mixed term $\langle f_j(0) - f_0, \tilde{f}_j(0) - \tilde{f}_0 \rangle_{\text{pot}}$, and since \mathcal{H} is preserved, (8.2) implies that

$$\mathcal{H}(f_j(t_j), \tilde{f}_j(t_j)) = \mathcal{H}(f_j(0), \tilde{f}_j(0)) \rightarrow \mathcal{H}(f_0, \tilde{f}_0).$$

Since $(f_j(t_j), \tilde{f}_j(t_j)) \in \mathcal{F}_M$ this means that $(f_j(t_j), \tilde{f}_j(t_j))$ is a minimizing sequence for \mathcal{H} in \mathcal{F}_M . By Theorem 2.1 there exists a subsequence such that up to spatial shifts,

$$\|f_j(t_j) - f_0\|_{\text{pot}} + \|\tilde{f}_j(t_j) - \tilde{f}_0\|_{\text{pot}} \rightarrow 0;$$

at this point the uniqueness assumption for (f_0, \tilde{f}_0) enters. Again by Lemma 3.4 this is true also for the mixed term $\langle f_j(t_j) - f_0, \tilde{f}_j(t_j) - \tilde{f}_0 \rangle_{\text{pot}}$, and (8.2) with $(f, \tilde{f}) = (f_j(t_j), \tilde{f}_j(t_j))$ implies that also $d((f_j(t_j), \tilde{f}_j(t_j)), (f_0, \tilde{f}_0)) \rightarrow 0$. Hence all three terms in (8.5) converge to zero, which is a contradiction.

9. Appendix: Facts on the Decoupled Minimizers

Here we establish the claims on the decoupled variational problems referred to in Sect. 4. Several of these claims, in particular for the non-flat case, can be found in the literature. *Existence.* In most of the previous investigations the existence of minimizers in the decoupled cases was not done exactly for the problems stated in Sect. 4: Either the Casimir functional, which in the case at hand corresponds to the $L^{1+1/k}$ norm, was included into the functional to be minimized, i.e., an energy-Casimir functional instead of the energy was minimized, and only the constraint on the L^1 norm was posed, or the energy was minimized under the constraint that the sum of the mass and the Casimir functional was fixed. In the three dimensional case the problem with two constraints in the form stated in Sect. 4 has been dealt with in [2, 29]. We briefly show how the method used in [2] can be adapted to the flat case.

With the help of the Riesz rearrangement inequality [20, 3.7] and the fact that the kinetic energy as well as the constraints are invariant under spherically symmetric rearrangements, the problem is reduced to minimizing $\mathcal{H}(\tilde{f})$, where the functions \tilde{f} in the constraint set have the form

$$\tilde{f}(\tilde{x}, \tilde{v}) = \varphi(|\tilde{x}|, |\tilde{v}|),$$

with $\varphi : [0, \infty]^2 \rightarrow [0, \infty[$ non-increasing in each argument. This monotonicity implies that for $1 \leq q \leq 1 + 1/\tilde{k}$,

$$\begin{aligned} \tilde{f}^q(\tilde{x}, \tilde{v})|\tilde{x}|^2|\tilde{v}|^2 &\leq C \int_0^{|\tilde{x}|} \int_0^{|\tilde{v}|} \varphi^q(r, s)rs \, ds \, dr \leq C \|\tilde{f}\|_q^q, \\ \tilde{f}(\tilde{x}, \tilde{v})|\tilde{x}|^2|\tilde{v}|^4 &\leq C \int_0^{|\tilde{x}|} \int_0^{|\tilde{v}|} \varphi(r, s)rs^3 \, ds \, dr \leq C E_{\text{kin}}(\tilde{f}). \end{aligned}$$

Hence

$$\tilde{f}(\tilde{x}, \tilde{v}) \leq g(\tilde{x}, \tilde{v}) := C \begin{cases} |\tilde{x}|^{-2/q}|\tilde{v}|^{-2/q}, & \text{for } |\tilde{v}| \leq V(|\tilde{x}|), \\ |\tilde{x}|^{-2}|\tilde{v}|^{-4}, & \text{for } |\tilde{v}| > V(|\tilde{x}|), \end{cases}$$

where $V(|\tilde{x}|) > 0$ is an arbitrary function and the constant C depends on $E_{\text{kin}}(\tilde{f})$, $\|\tilde{f}\|_1$, and $\|\tilde{f}\|_{1+1/\tilde{k}}$, quantities which are bounded along minimizing sequences. The function g induces the spatial density

$$\begin{aligned} \rho_g(\tilde{x}) &= C|\tilde{x}|^{-2/q} \int_0^{V(|\tilde{x}|)} |\tilde{v}|^{1-2/q} d|\tilde{v}| + C|\tilde{x}|^{-2} \int_{V(|\tilde{x}|)}^\infty |\tilde{v}|^{-3} d|\tilde{v}| \\ &= C|\tilde{x}|^{-2/q} V^{2-2/q}(|\tilde{x}|) + C|\tilde{x}|^{-2} V^{-2}(|\tilde{x}|). \end{aligned}$$

The choice

$$V(|\tilde{x}|) = V_q(|\tilde{x}|) := |\tilde{x}|^{(1-q)/(2q-1)}$$

yields the estimate

$$\rho_g(\tilde{x}) \leq C |\tilde{x}|^{-2q/(2q-1)} \tag{9.1}$$

with the exponent $s := -2q/(2q - 1)$ being such that

$$s < -3/2 \text{ for } 1 < q < 3/2, \quad s > -3/2 \text{ for } q > 3/2.$$

We split the estimate for \tilde{f} by choosing $q = 1 + 1/\tilde{k} > 3/2$ for $|\tilde{x}| \leq 1$ and $q \in]1, 4/3[$ for $|\tilde{x}| > 1$ so that

$$\tilde{f}(\tilde{x}, \tilde{v}) \leq g(\tilde{x}, \tilde{v}) := C \begin{cases} |\tilde{x}|^{-2/(1+1/\tilde{k})} |\tilde{v}|^{-2/(1+1/\tilde{k})} & \text{for } |\tilde{x}| \leq 1 \wedge |\tilde{v}| \leq V_{1+1/\tilde{k}}(|\tilde{x}|), \\ |\tilde{x}|^{-2} |\tilde{v}|^{-4} & \text{for } |\tilde{x}| \leq 1 \wedge |\tilde{v}| > V_{1+1/\tilde{k}}(|\tilde{x}|), \\ |\tilde{x}|^{-2/q} |\tilde{v}|^{-2/q} & \text{for } |\tilde{x}| > 1 \wedge |\tilde{v}| \leq V_q(|\tilde{x}|), \\ |\tilde{x}|^{-2} |\tilde{v}|^{-4} & \text{for } |\tilde{x}| > 1 \wedge |\tilde{v}| > V_q(|\tilde{x}|). \end{cases}$$

By (9.1) we can obtain exponents $s_1 > -8/5$ and $s_2 < -8/5$ such that

$$\rho_{\tilde{f}}(\tilde{x}) \leq \rho_g(\tilde{x}) \leq \begin{cases} Cr^{s_1} & \text{for } |\tilde{x}| \leq 1, \\ Cr^{s_2} & \text{for } |\tilde{x}| > 1. \end{cases}$$

By the Hardy-Littlewood-Sobolev inequality this implies that g has finite potential energy. The crucial step in the existence proof for a minimizer is the convergence of the potential energy along a minimizing sequence (\tilde{f}_j) . Since $0 \leq \tilde{f}_j \leq g$, the finiteness of the potential energy for g allows us to pass to the limit using the dominated convergence theorem.

Saturation of the constraints. Minimizers of the decoupled problems always saturate the constraints, i.e., $\|f_0^{3D}\|_1 = M$, $\|f_0^{3D}\|_{1+1/k} = N$, and similarly for the flat case, because if for a minimizer one (or both) equalities were replaced by strict inequalities, then this minimizer can be rescaled in such a way that the constraints become saturated but the energy strictly decreases, which is a contradiction. A similar argument was used in the proof of Proposition 7.1.

The Euler-Lagrange relation and symmetry. For minimizers of the flat or non-flat problem the phase space distributions are functions of the local energy, more precisely, they satisfy relations as stated in Theorem 7.2, the only differences being that

$$E(x, v) := \frac{1}{2}|v|^2 + U_0^{3D}(x), \quad E(\tilde{x}, \tilde{v}) := \frac{1}{2}|\tilde{v}|^2 + U_0^{FL}(\tilde{x}),$$

and the interaction term $\int U_0^{3D} \rho_0^{FL}$ in the definitions of E_0 and \tilde{E}_0 is dropped.

The asserted symmetry of the minimizers follows from the fact that symmetric decreasing rearrangements in x or \tilde{x} strictly decrease the energy except when ρ_0^{3D} and ρ_0^{FL} and hence also the induced potentials are symmetric with respect to some point, cf. [20, 3.7, 3.9].

Virial identity and compact support. Both flat and non-flat minimizers satisfy the virial identity

$$2E_{\text{kin}}(f_0^{FL}) = -E_{\text{pot}}(f_0^{FL}), \quad 2E_{\text{kin}}(f_0^{3D}) = -E_{\text{pot}}(f_0^{3D}).$$

This follows from the fact that these minimizers are time-independent solutions of the corresponding Vlasov-Poisson systems. A direct proof based on their minimizing property and scaling is given in [2, 3.2]. The virial identities together with the restrictions on k and \tilde{k} immediately imply that the cut-off energies E_0 and \tilde{E}_0 in the Euler-Lagrange relations are strictly negative. In order to show that the minimizers have compact support it therefore suffices to show that their potentials converge to zero at spatial infinity. We indicate the corresponding argument for the 3D case, the flat one being completely analogous. Applying the Hölder inequality to the first term in the estimate

$$-U_0^{3D}(x) \leq \int_{|x-y| \leq R} \frac{\rho_0^{3D}(y)}{|x-y|} dy + \frac{M}{R}, \quad R > 0$$

implies that the potential is bounded and vanishes at spatial infinity, provided $\rho_0^{3D} \in L^p(\mathbb{R}^3)$ with $p > 3/2$. While a-priori this need not be the case for $0 < k < 7/2$ we can use the fact that similar to (7.3),

$$\rho_0^{3D} = c(E_0 - U_0^{3D})_+^n,$$

start with the known integrability $U_0^{3D} \in L^6(\mathbb{R}^3)$ to conclude that $\rho_0^{3D} \in L^{6/n}(\mathbb{R}^3)$, and obtain a new integrability estimate for the potential through the weak Young inequality. After finitely many iterations the desired integrability of ρ_0^{3D} follows, cf. [27, Prop. 2.7].

Uniqueness for the 3D problem. First we notice that by the virial relation the Lagrange multipliers are uniquely determined by the constraint parameters M and N . In the 3D case we can now continue as follows. A minimizer is completely determined by its potential. The latter satisfies the Emden-Fowler equation

$$\frac{1}{r^2}(r^2 U')' = c(E_0 - U)_+^n, \text{ i.e., } \frac{1}{r^2}(r^2 y')' = -cy_+^n,$$

where $y = E_0 - U$, and the constant c depends only on k and M, N . The solutions of this equation which are regular at the origin are uniquely determined by their value at the origin. Moreover, the scaling $y_\alpha(r) = \alpha y(\alpha^\lambda r)$ with $\lambda = (k + 1/2)/2$ turns solutions into solutions. But the mass constraint fixes this scaling, and uniqueness of the minimizer follows. For more details we refer to [27, p. 464]. Unfortunately, there is no analogue to the Emden-Fowler equation in the flat case—the flat potential does not satisfy the Poisson equation—and uniqueness in the flat case is not known.

The radius relation (4.1) in the 3D case. Each minimizer is a spherically symmetric steady state (f, ρ, U) of the three dimensional Vlasov-Poisson system, and for each choice of M and N there is a unique such steady state. If (f, ρ, U) is a steady state and $\alpha, \beta > 0$ are arbitrary, then

$$f_{\alpha\beta}(x, v) = \alpha^2 \beta f(\alpha x, \beta v), \quad \rho_{\alpha\beta}(x) = \alpha^2 \beta^{-2} \rho(\alpha x), \quad U_{\alpha\beta}(x) = \beta^{-2} U(\alpha x)$$

defines another one. There is a unique steady state (f^*, ρ^*, U^*) with $\|f^*\|_1 = 1 = \|f^*\|_{1+1/k}$, and the minimizer with general M and N is obtained by rescaling f^* with the parameters

$$\alpha = M^{(1-2k)/3} N^{(2k+2)/3}, \quad \beta = M^{(k-2)/3} N^{-(k+1)/3}.$$

Since $R = R^*/\alpha$ this implies (4.1).

The radius relation (4.2) in the flat case. We do not know whether for each choice of M and N there exists a unique minimizer f_0^{FL} , and so we cannot use the argument above to prove (4.2).

To obtain a two-parameter family of minimizers which obeys the radius relation (4.2) we proceed as follows. Since minimizers a-posteriori saturate the constraints we redefine

$$\mathcal{F}_{M,N}^{FL} := \left\{ \tilde{f} \in L^1_+(\mathbb{R}^4) \mid \|\tilde{f}\|_1 = M, \|\tilde{f}\|_{1+1/\tilde{k}} = N, E_{\text{kin}}(\tilde{f}) < \infty \right\}.$$

For $\tilde{f} \in \mathcal{F}_{1,1}^{FL}$ we define the rescaled function

$$\tilde{f}_{\mu,\nu}(\tilde{x}, \tilde{v}) := \mu \tilde{f}(\mu \tilde{x}, \nu \tilde{v}).$$

Then

$$\|\tilde{f}_{\mu,\nu}\|_1 = \mu^{-1} \nu^{-2}, \quad \|\tilde{f}_{\mu,\nu}\|_{1+1/\tilde{k}} = \mu^{(1-\tilde{k})/(1+\tilde{k})} \nu^{-2\tilde{k}/(\tilde{k}+1)}, \quad \mathcal{H}(\tilde{f}_{\mu,\nu}) = \mu^{-1} \nu^{-4} \mathcal{H}(\tilde{f}).$$

For $M, N > 0$ arbitrary we choose μ, ν such that $\tilde{f}_{\mu, \nu} \in \mathcal{F}_{M, N}^{\text{FL}}$, i.e.,

$$\mu = M^{-\tilde{k}} N^{\tilde{k}+1}, \quad \nu = M^{(\tilde{k}-1)/2} N^{-(\tilde{k}+1)/2}.$$

The mapping $\mathcal{F}_{1,1}^{\text{FL}} \ni \tilde{f} \mapsto \tilde{f}_{\mu, \nu} \in \mathcal{F}_{M, N}^{\text{FL}}$ is one-to-one and onto. Let f^{FL} denote an arbitrary, spherically symmetric minimizer of \mathcal{H} over the set $\mathcal{F}_{1,1}^{\text{FL}}$. Then $f_{\mu, \nu}^{\text{FL}}$ is a minimizer of \mathcal{H} over $\mathcal{F}_{M, N}^{\text{FL}}$ which we denote by $f_{M, N}^{\text{FL}}$. To see this we observe that any function $\tilde{g} \in \mathcal{F}_{M, N}^{\text{FL}}$ can be written as $\tilde{g} = \tilde{f}_{\mu, \nu}$ where $\tilde{f} \in \mathcal{F}_{1,1}^{\text{FL}}$, and hence

$$\mathcal{H}(\tilde{g}) = \mathcal{H}(\tilde{f}_{\mu, \nu}) = \mu^{-1} \nu^{-4} \mathcal{H}(\tilde{f}) \geq \mu^{-1} \nu^{-4} \mathcal{H}(f^{\text{FL}}) = \mathcal{H}(f_{\mu, \nu}^{\text{FL}}).$$

If \tilde{R}^* denotes the radius of the spatial support of f^{FL} , then the spatial support of $f_{M, N}^{\text{FL}}$ has radius

$$\tilde{R} = \mu^{-1} \tilde{R}^* = \tilde{R}^* M^{\tilde{k}} N^{-(\tilde{k}+1)},$$

and this is the remaining assertion (4.2).

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