

A non-variational approach to nonlinear stability in stellar dynamics applied to the King model

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Abstract

In previous work by Y. Guo and G. Rein, nonlinear stability of equilibria in stellar dynamics, i.e., of steady states of the Vlasov-Poisson system, was accessed by variational techniques. Here we propose a different, non-variational technique and use it to prove nonlinear stability of the King model against a class of spherically symmetric, dynamically accessible perturbations. This model is very important in astrophysics and was out of reach of the previous techniques.

1 Introduction

In astrophysics a galaxy or a globular cluster is often modeled as a large ensemble of particles, i.e., stars, which interact only by the gravitational field which they create collectively, collisions among the stars being sufficiently

rare to be neglected. The time evolution of the distribution function $f = f(t, x, v) \geq 0$ of the stars in phase space is then given by the Vlasov-Poisson system:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \quad (1.1)$$

$$\Delta U = 4\pi\rho, \quad \lim_{|x| \rightarrow \infty} U(t, x) = 0, \quad (1.2)$$

$$\rho(t, x) = \int f(t, x, v) dv. \quad (1.3)$$

Here $t \in \mathbb{R}$ denotes time, and $x, v \in \mathbb{R}^3$ denote position and velocity, $U = U(t, x)$ is the gravitational potential of the ensemble, and $\rho = \rho(t, x)$ is its spatial density.

A well-known approach to obtaining steady states of the Vlasov-Poisson system is to make an ansatz of the form

$$f_0(x, v) = \phi(E),$$

where the particle energy E is defined as

$$E := \frac{1}{2}|v|^2 + U_0(x).$$

Since for a time-independent potential $U_0 = U_0(x)$ the particle energy is conserved along the characteristics of the Vlasov equation (1.1) it remains to make sure that for the chosen ϕ the resulting semilinear Poisson equation

$$\Delta U_0 = 4\pi \int \phi \left(\frac{1}{2}|v|^2 + U_0 \right) dv \quad (1.4)$$

has a solution. For a very large class of ansatz functions ϕ this approach leads to steady states with finite mass and compact support, cf. [RR]. Steady states where as above the distribution of the particles in phase space depends only on the particle energy are usually called *isotropic* and are always spherically symmetric. The latter follows by applying the results in [GNN] to Eqn. (1.4), cf. also [R1, Thm. 3]. A central question in stellar dynamics, which has attracted considerable attention in the astrophysics literature, cf. [BT, FP] and the references there, is the dynamical stability of such steady states.

The Vlasov-Poisson system is conservative, i.e., the total energy

$$\mathcal{H}(f) := E_{\text{kin}}(f) + E_{\text{pot}}(f) = \frac{1}{2} \int |v|^2 f(x, v) dv dx - \frac{1}{8\pi} \int |\nabla U_f(x)|^2 dx$$

of a state f is conserved along solutions and hence is a natural candidate for a Lyapunov function in a stability analysis; U_f denotes the potential induced by f . However, the energy does not have critical points, i.e., the linear part in an expansion about any state f_0 with potential U_0 does not vanish:

$$\mathcal{H}(f) = \mathcal{H}(f_0) + \iint \left(\frac{1}{2} |v|^2 + U_0 \right) (f - f_0) dv dx - \frac{1}{8\pi} \int |\nabla U_f - \nabla U_0|^2 dx.$$

To remedy this situation one observes that for any reasonable function Φ the so-called *Casimir functional*

$$\mathcal{C}(f) := \iint \Phi(f(x, v)) dv dx$$

is conserved as well. If the energy-Casimir functional

$$\mathcal{H}_C := \mathcal{H} + \mathcal{C}$$

is expanded about an isotropic steady state, then

$$\begin{aligned} \mathcal{H}_C(f) &= \mathcal{H}_C(f_0) + \iint (E + \Phi'(f_0))(f - f_0) dv dx \\ &\quad - \frac{1}{8\pi} \int |\nabla U_f - \nabla U_0|^2 dx + \frac{1}{2} \iint \Phi''(f_0)(f - f_0)^2 dv dx + \dots \\ &=: \mathcal{H}_C(f_0) + D\mathcal{H}_C(f_0)[f - f_0] + D^2\mathcal{H}_C(f_0)[f - f_0] + \dots \end{aligned}$$

Now one can try to choose Φ in such a way that at least formally f_0 is a critical point of the energy-Casimir functional, i.e., $\Phi'(f_0) = -E$. If $\phi' < 0$ on the support of f_0 the former relation holds on this support if $\Phi' = -\phi^{-1}$. The formal second order variation in the above expansion then takes the form

$$D^2\mathcal{H}_C(f_0)[g] = \frac{1}{2} \iint_{\{f_0 > 0\}} \frac{1}{-\phi'(E)} g^2 dv dx - \frac{1}{8\pi} \int |\nabla_x U_g|^2 dx. \quad (1.5)$$

It is natural to expect that positive definiteness of this quadratic form should imply stability for f_0 . Ever since the seminal work of ANTONOV [An] there have been vigorous efforts in the astrophysics community to establish this positive definiteness and to derive stability results in this way.

An important step in this direction was to show that the above quadratic form is positive definite on linearized, dynamically accessible perturbations.

To make this precise we define the Lie-Poisson bracket of two functions f_1, f_2 of x and v as

$$\{f_1, f_2\} := \nabla_x f_1 \cdot \nabla_v f_2 - \nabla_v f_1 \cdot \nabla_x f_2. \quad (1.6)$$

Then the following holds:

Lemma 1.1 *Let $\phi' < 0$ and let $h \in C_c^\infty(\mathbb{R}^6)$ be spherically symmetric with support in the set $\{f_0 > 0\}$ and such that $h(x, -v) = -h(x, v)$. Then*

$$D^2\mathcal{H}_C(f_0)[\{f_0, h\}] \geq -\frac{1}{2} \int_{f_0 > 0} \phi'(E) \left[|x \cdot v|^2 \left| \left\{ E, \frac{h}{x \cdot v} \right\} \right|^2 + \frac{1}{r} U'_0 h^2 \right] dv dx.$$

Here U'_0 denotes the radial derivative of the steady state potential. Since U_0 is radially increasing, the right hand side in the estimate above is indeed positive for $h \neq 0$. We refer to [KS, SDLP] for astrophysical investigations where this result is used to analyze linearized stability. We do not go into the reasons why perturbations of the form $\{f_0, h\}$ are called dynamically accessible for the linearized system, but the interested reader can find background on this and related concepts in [M]. For the sake of completeness we provide a proof of this elegant result in the Appendix. Despite its significance the result is still quite a distance away from a true, nonlinear stability result. There are at least two serious mathematical difficulties. Firstly, it is very challenging to use the positivity of $D^2\mathcal{H}_C(f_0)[g]$ to control the higher order remainder in the expansion of the energy-Casimir functional [Wa]. This is due to the non-smooth nature of $f_0 = \phi(E)$ in all important examples. Secondly, even if one succeeds in controlling the higher order terms, the positivity of $D^2\mathcal{H}_C(f_0)[g]$ in the lemma is only valid for certain perturbation of the form $g = \{f_0, h\}$. This class of perturbations is invariant under solutions of the linearized Vlasov-Poisson system, but it is not invariant under solutions of the nonlinear system.

To overcome these difficulties a variational approach was initiated by WOLANSKY [Wo1] and then developed systematically by GUO and REIN [G1, G2, GR1, GR2, GR3, GR4, R2, R3, RG]. Their method entirely avoids the delicate analysis of the second order term $D^2\mathcal{H}_C(f_0)$ in (1.5), and it has led to the first rigorous nonlinear stability proofs for a large class of steady states. More precisely, a large class of steady states is obtained as minimizers of energy-Casimir functionals under a mass constraint $\int f = M$, and their minimizing property then entails their stability. In particular, all polytropes $f_0(x, v) = (E_0 - E)_+^k$ with $0 < k \leq 7/2$ are covered; here $E_0 < 0$ is

a certain cut-off energy, and $(\cdot)_+$ denotes the positive part. For $k > 7/2$ the corresponding steady state has infinite mass and is therefore unphysical. In addition, many new stable galaxy models were established. The variational method has also been investigated in [DSS, H, LMR, SS, Wo2].

Despite its considerable success, the variational approach has drawbacks and limitations, the main one being that by its very nature it can not access the stability of steady states which are only local, but not global minimizers of the energy-Casimir functional. Since the existence of the steady state as a (global) minimizer is aimed for, certain growth conditions on the Casimir function Φ are needed, which are not satisfied for all steady states with $\phi' < 0$. Most notably, the King model obtained by

$$f_0(x, v) = (e^{E_0 - E} - 1)_+$$

is the single most important model which is currently out of the reach. It describes isothermal galaxies and is widely used in astrophysics. The corresponding Casimir function (2.2) has very slow growth for $f \rightarrow \infty$, and as a result the variational method fails.

The aim of the present paper is to develop a new approach to nonlinear stability results for steady states which need not be global minimizers of the corresponding energy-Casimir functional by exploiting Lemma 1.1. Although we are aiming for a general approach, we focus here on the King model and as a first step establish its nonlinear stability against spherically symmetric, dynamically accessible perturbations. An application of the present approach to the relativistic Vlasov-Poisson system for which the variational approach does not apply is given in [HR].

The paper proceeds as follows. In the next section we formulate our results. The nonlinear stability of the King model is an easy corollary of the following main theorem: In a certain neighborhood of the King model the potential energy distance of a perturbation can be controlled in terms of the energy-Casimir distance. In particular, within a certain class of perturbations, which is invariant under solutions of the nonlinear Vlasov-Poisson system, the King model is a local minimizer of the corresponding energy-Casimir functional. The resulting stability estimate is more explicit than the ones obtained by the variational approach. The main part of the work is then done in Section 3 where the local minimizing property of the King model is established. In an appendix we give a proof of Lemma 1.1.

2 Main results

We start with a steady state f_0 with induced potential U_0 and spatial density ρ_0 , satisfying the relation

$$f_0(x, v) = \phi_0(E) := (e^{E_0 - E} - 1)_+, \quad E := \frac{1}{2}|v|^2 + U_0(x). \quad (2.1)$$

The cut-off energy $E_0 < 0$ is a given negative constant and $(\cdot)_+$ denotes the positive part. The corresponding Casimir function in the sense of the introduction is

$$\Phi_0(f) := (1 + f) \ln(1 + f) - f. \quad (2.2)$$

The existence of such King models, i.e., of suitable solutions of the resulting semilinear Poisson equation (1.4), is established in [RR]. Such a model has compact support

$$\text{supp } f_0 = \{(v, v) \in \mathbb{R}^6 \mid E(x, v) \leq E_0\} =: \{E \leq E_0\},$$

and it is spherically symmetric. A state f is called *spherically symmetric* if for any rotation $A \in \text{SO}(3)$,

$$f(x, v) = f(Ax, Av), \quad x, v \in \mathbb{R}^3.$$

It is well known that non-negative, smooth, and compactly supported initial data $f(0) \in C_c^1(\mathbb{R}^6)$ launch unique global smooth solutions $t \mapsto f(t)$ of the Vlasov-Poisson system [Pf, LP, Sch]. If the initial datum is spherically symmetric then this symmetry is preserved, and the modulus of the particle angular momentum squared,

$$L := |x \times v|^2 = |x|^2|v|^2 - (x \cdot v)^2,$$

is conserved along characteristics of the Vlasov equation. Hence for any smooth function Φ such that $\Phi(0, L) = 0$, $L \geq 0$, the functional

$$\iint \Phi(f, L) dv dx$$

is conserved along spherically symmetric solutions of the Vlasov-Poisson system; unless explicitly stated otherwise integrals \int always extend over \mathbb{R}^3 .

We consider the following class of perturbations:

$$\mathcal{S}_{f_0} := \left\{ f \in L^1(\mathbb{R}^6) \mid f \geq 0 \text{ spherically symmetric,} \right. \\ \left. \iint \Phi(f, L) = \iint \Phi(f_0, L) \text{ for all } \Phi \in C^2([0, \infty]^2) \text{ with} \right. \\ \left. \Phi(0, L) = \partial_f \Phi(0, L) = 0, L \geq 0, \text{ and } \partial_f^2 \Phi \text{ bounded} \right\}.$$

As noted above, the class $\mathcal{S}_{f_0} \cap C_c^1(\mathbb{R}^6)$ is invariant under solutions of the Vlasov-Poisson system. Moreover, functions in \mathcal{S}_{f_0} are equi-measurable to f_0 , i.e., for every $\tau > 0$ the sets $\{f > \tau\}$ and $\{f_0 > \tau\}$ have the same measure, in particular, $\|f\|_p = \|f_0\|_p$ for any L^p -norm, $p \in [1, \infty]$.

For the Casimir function Φ_0 defined in (2.2) we define the energy-Casimir functional as in the introduction. Then for $f \in \mathcal{S}_{f_0}$ we have

$$\mathcal{H}_C(f) - \mathcal{H}_C(f_0) = \iint [\Phi_0(f) - \Phi_0(f_0) + (E - E_0)(f - f_0)] dv dx \\ - \frac{1}{8\pi} \int |\nabla U_f - \nabla U_0|^2 dx;$$

notice that $\int f = \int f_0$ which allows us to bring in the term $E_0(f - f_0)$. Now

$$(E - E_0)(f - f_0) \geq -\Phi'_0(f_0)(f - f_0)$$

with equality on the support of f_0 , and hence for $f \in \mathcal{S}_{f_0}$,

$$\Phi_0(f) - \Phi_0(f_0) + (E - E_0)(f - f_0) \geq \frac{1}{2} \inf_{0 \leq \tau \leq \|f_0\|_\infty} \Phi''_0(\tau) (f - f_0)^2 \\ \geq C_0 (f - f_0)^2 \quad (2.3)$$

where $C_0 := 1/(2 + 2\|f_0\|_\infty)$; notice again that $\|f\|_\infty = \|f_0\|_\infty$ for any $f \in \mathcal{S}_{f_0}$. The deviation from the steady state is going to be measured by the quantity

$$d(f, f_0) := \iint [\Phi_0(f) - \Phi_0(f_0) + (E - E_0)(f - f_0)] dv dx \\ + \frac{1}{8\pi} \int |\nabla U_f - \nabla U_0|^2 dx, \quad (2.4)$$

which, as we have seen, controls both $\|f - f_0\|_2$ and $\|\nabla U_f - \nabla U_0\|_2$, and satisfies the following relation to the energy-Casimir functional:

$$d(f, f_0) = \mathcal{H}_C(f) - \mathcal{H}_C(f_0) + \frac{1}{4\pi} \int |\nabla U_f - \nabla U_0|^2 dx. \quad (2.5)$$

Our stability result is the following:

Theorem 2.1 *There exist constants $\delta > 0$ and $C > 0$ such that for any solution $t \mapsto f(t)$ of the Vlasov-Poisson system with $f(0) \in \mathcal{S}_{f_0} \cap C_c^1(\mathbb{R}^6)$ and*

$$d(f(0), f_0) \leq \delta$$

the estimate

$$d(f(t), f_0) \leq C d(f(0), f_0)$$

holds for all time $t > 0$.

Remark: In order to better understand the perturbation class \mathcal{S}_{f_0} we show that it contains *spherically symmetric, dynamically accessible perturbations* by which we mean the following: Let $H = H(x, v) \in C^2(\mathbb{R}^6)$ be spherically symmetric, and let $g = g(s, x, v)$ denote the solution of the linear problem

$$\partial_s g + \nabla_v H \cdot \nabla_x g - \nabla_x H \cdot \nabla_v g = 0, \quad \text{i.e.,} \quad \partial_s g(s) = \{H, g(s)\},$$

with initial datum $g(0) = f_0$; we assume that H is such that this solution exists on some interval I about $s = 0$. Then for any $s \in I$, $g(s)$ is spherically symmetric and equi-measurable with f_0 . Moreover, for any function Φ as considered in the definition of \mathcal{S}_{f_0} ,

$$\partial_s \Phi(g(s), L) = \partial_f \Phi(g(s), L) \{H, g(s)\},$$

and by a simple computation, $\{H, L\} = 0$. Hence

$$\begin{aligned} \frac{d}{ds} \iint \Phi(g(s), L) dv dx &= \iint (\partial_f \Phi \{H, g(s)\} + \partial_L \Phi \{H, L\}) dv dx \\ &= \iint \{H, \Phi(g(s), L)\} dv dx = 0 \end{aligned}$$

after an integration by parts. This means that $g(s) \in \mathcal{S}_{f_0}$ for any such generating function H and any s . The only undesirable restriction in the class \mathcal{S}_{f_0} or in the generating functions H respectively is the spherical symmetry which hopefully can be removed in the future.

The stability result Theorem 2.1 is easily deduced from the following theorem:

Theorem 2.2 *There exist constants $\delta_0 > 0$, and $C_0 > 0$ such that for all $f \in \mathcal{S}_{f_0}$ with $d(f, f_0) \leq \delta_0$ the following estimate holds:*

$$\mathcal{H}_C(f) - \mathcal{H}_C(f_0) \geq C_0 \|\nabla U_f - \nabla U_0\|_2^2.$$

Before going into the proof of this theorem, which will occupy the rest of this paper, we conclude this section by deducing our stability result from it.

Proof of Theorem 2.1. Let $\delta := \delta_0(1 + 1/(4\pi C_0))^{-1}$ with δ_0 and C_0 from Theorem 2.2. Consider a solution $t \mapsto f(t)$ of the Vlasov-Poisson system with $f(0) \in \mathcal{S}_{f_0} \cap C_c^1(\mathbb{R}^6)$ and

$$d(f(0), f_0) \leq \delta < \delta_0.$$

Then by continuity we can choose some maximal $t^* \in]0, \infty]$ such that

$$d(f(t), f_0) < \delta_0, \quad t \in [0, t^*[.$$

Now $f(t) \in \mathcal{S}_{f_0}$ for all t , and hence Theorem 2.2, the relation (2.5) of d to the energy-Casimir functional, and the fact that the latter is a conserved quantity yield the following chain of estimates for $t \in [0, t^*[$:

$$\begin{aligned} d(f(t), f_0) &= \mathcal{H}_C(f(t)) - \mathcal{H}_C(f_0) + \frac{1}{4\pi} \|\nabla U_{f(t)} - \nabla U_{f_0}\|_2^2 \\ &\leq \mathcal{H}_C(f(t)) - \mathcal{H}_C(f_0) + \frac{1}{4\pi C_0} (\mathcal{H}_C(f(t)) - \mathcal{H}_C(f_0)) \\ &= \left(1 + \frac{1}{4\pi C_0}\right) (\mathcal{H}_C(f(t)) - \mathcal{H}_C(f_0)) \\ &\leq \left(1 + \frac{1}{4\pi C_0}\right) d(f(0), f_0) < \delta_0. \end{aligned}$$

This implies that $t^* = \infty$, and Theorem 2.1 is established. \square

3 Proof of Theorem 2.2

Theorem 2.2 is proven by contradiction. There are two main ingredients: The first part (Subsection 3.1) is a general argument to establish that if the estimate in the theorem fails, then there exists a non-zero function g such that $D^2\mathcal{H}_C(f_0)[g] \leq 0$. The second (Subsection 3.2) is to use the measure-preserving property incorporated in our perturbation class \mathcal{S}_{f_0} to conclude that $g = \{f_0, h\}$ for some function h . This leads to a contradiction to Lemma 1.1 (Subsection 3.3).

3.1 Existence of $g \neq 0$ with $D^2\mathcal{H}_C(f_0)[g] \leq 0$

The aim of this subsection is to prove the following result:

Lemma 3.1 *Assume that Theorem 2.2 were false. Then there is a function $g \in L^2(\mathbb{R}^6)$ which is spherically symmetric, supported in the set $\{E \leq E_0\}$, even in v , i.e., $g(x, -v) = g(x, v)$, and such that*

$$\frac{1}{8\pi} \|\nabla U_g\|_2^2 = 1, \quad (3.1)$$

$$D^2\mathcal{H}_C(f_0)[g] = \frac{1}{2} \iint_{\{f_0 > 0\}} \Phi_0''(f_0) g^2 dv dx - 1 \leq 0, \quad (3.2)$$

$$\iint \partial_f \Phi(f_0, L) g dv dx = 0 \quad (3.3)$$

for all functions Φ as specified in the definition of \mathcal{S}_{f_0} .

Proof. If Theorem 2.2 were false, then for any $n \in \mathbb{N}$ there exists $f_n \in \mathcal{S}_{f_0}$ such that

$$d(f_n, f_0) < \frac{1}{n}$$

but

$$\mathcal{H}_C(f_n) - \mathcal{H}_C(f_0) < \frac{1}{8\pi n} \|\nabla U_{f_n} - \nabla U_0\|_2^2, \quad (3.4)$$

in particular, $f_n \neq f_0$. We define

$$\sigma_n := \frac{1}{\sqrt{8\pi}} \|\nabla U_{f_n} - \nabla U_0\|_2, \quad g_n := \frac{1}{\sigma_n} (f_n - f_0) \quad (3.5)$$

so that

$$\frac{1}{8\pi} \|\nabla U_{g_n}\|_2^2 = 1 \quad (3.6)$$

and $f_n = f_0 + \sigma_n g_n$. By (2.5) and (3.5),

$$\sigma_n^2 \leq d(f_n, f_0) < \frac{1}{n}. \quad (3.7)$$

Bounds on (g_n) and weak limit.

As a first step in the proof of the lemma we need to establish bounds on the sequence (g_n) which allow us extract a subsequence that converges to some g which will be our candidate for the function asserted in the lemma. By

(3.5), (2.4), (2.5), and (3.4) we find that

$$\begin{aligned}
& \frac{1}{\sigma_n^2} \iint [\Phi_0(f_0 + \sigma_n g_n) - \Phi_0(f_0) + (E - E_0) \sigma_n g_n] dv dx - 1 \\
&= \frac{1}{\sigma_n^2} \left(d(f_n, f_0) - \frac{1}{4\pi} \int |\nabla U_{f_n} - \nabla U_0|^2 dx \right) \\
&= \frac{1}{\sigma_n^2} (\mathcal{H}_C(f_n) - \mathcal{H}_C(f_0)) < \frac{1}{\sigma_n^2} \frac{1}{8\pi n} \|\nabla U_{f_n} - \nabla U_0\|_2^2 = \frac{1}{n}. \quad (3.8)
\end{aligned}$$

Recalling the estimate (2.3) which is applicable since $f_n \in \mathcal{S}_{f_0}$ this implies that

$$1 + \frac{1}{n} > \frac{1}{\sigma_n^2} \iint [\dots] dv dx \geq C_0 \frac{1}{\sigma_n^2} \iint |f_n - f_0|^2 dv dx = C_0 \iint |g_n|^2 dv dx,$$

which means that the sequence (g_n) is bounded in $L^2(\mathbb{R}^6)$. Moreover, since the integrand [...] in (3.8) is non-negative we find that

$$1 + \frac{1}{n} > \frac{1}{\sigma_n^2} \iint_{\{E \geq E_0\}} [\dots] dv dx \geq \frac{1}{\sigma_n} \iint_{\{E \geq E_0\}} (E - E_0) g_n dv dx,$$

where we used the fact that on the set $\{E \geq E_0\}$ the steady state distribution f_0 and hence also $\Phi_0(f_0)$ vanish while $\Phi_0(f_n) \geq 0$. By (3.7),

$$\iint_{\{E \geq E_0\}} (E - E_0) g_n dv dx \leq 2\sigma_n \rightarrow 0, \quad n \rightarrow \infty. \quad (3.9)$$

Now fix any $E_0 < E_1 < 0$. Since $\lim_{|x| \rightarrow \infty} U_0(x) = 0$ it follows that $E = E(x, v) > E_1$ for x or v large so that the set $\{E \leq E_1\} \subset \mathbb{R}^6$ is compact. Hence (g_n) is bounded in $L^1(\{E \leq E_1\})$. In addition

$$\iint_{\{E \geq E_1\}} g_n dv dx \leq \iint_{\{E \geq E_0\}} \frac{E - E_0}{E_1 - E_0} g_n dv dx \rightarrow 0; \quad (3.10)$$

notice that $g_n \geq 0$ outside the support of f_0 , i.e., on the set $\{E > E_0\}$. Thus we have shown that (g_n) is bounded in $L^1 \cap L^2(\mathbb{R}^6)$. We extract a subsequence, denoted again by (g_n) such that

$$g_n \rightharpoonup g \text{ weakly in } L^2(\mathbb{R}^6).$$

Since $g_n \geq 0$ on $\{E > E_0\}$ and since (3.10) holds for any $E_0 < E_1 < 0$ we conclude that g vanishes a. e. outside the set $\{E \leq E_0\}$ as desired. Since the functions g_n are spherically symmetric so is g .

Proof of (3.1).

In order to pass the weak convergence into (3.6) we need better bounds for the sequence (g_n) . Indeed, we can bound its kinetic energy:

$$\iint_{\{E \geq E_0\}} \left(\frac{1}{2}|v|^2 + U_0(x) - E_0 \right) g_n dv dx = \iint_{\{E \geq E_0\}} (E - E_0) g_n dv dx \rightarrow 0$$

by (3.9) so that

$$\iint_{\{E \geq E_0\}} \frac{1}{2}|v|^2 g_n dv dx$$

is bounded, while

$$\iint_{\{E \leq E_0\}} \frac{1}{2}|v|^2 |g_n| dv dx \leq (E_0 - U_0(0)) \iint_{\{E \leq E_0\}} |g_n| dv dx$$

is bounded as well; recall that U_0 is spherically symmetric and radially increasing. Now well known interpolation arguments imply that the sequence of induced spatial densities (ρ_{g_n}) is bounded in $L^1 \cap L^{7/5}(\mathbb{R}^3)$, cf. [R4, Ch. 1, Lemma 5.1], so without loss of generality,

$$\rho_{g_n} \rightharpoonup \rho_g \text{ weakly in } L^{7/5}(\mathbb{R}^3).$$

Let again $E_0 < E_1 < 0$ be arbitrary but fixed and choose $R_1 > 0$ such that $U_0(R_1) = E_1$ which implies that $E(x, v) \geq E_1$ for $|x| \geq R_1$. Then by (3.10),

$$\int_{\{|x| \geq R_1\}} |\rho_{g_n}| dx \leq \iint_{\{E \geq E_1\}} g_n dv dx \rightarrow 0.$$

The fact that the sequence (ρ_{g_n}) remains concentrated in this way gives the desired compactness:

$$\nabla U_{g_n} \rightarrow \nabla U_g \text{ strongly in } L^2(\mathbb{R}^3),$$

cf. [R4, Ch. 2, Lemma 3.2]. Hence we can pass to the limit in (3.6) and find that g satisfies the condition (3.1) in the lemma.

Proof of (3.2).

Since $\|g_n\|_2$ is bounded it follows from (3.7) that

$$\|\sigma_n g_n\|_2 \leq C \sigma_n \rightarrow 0.$$

Therefore, after extracting again a subsequence, $\sigma_n g_n \rightarrow 0$ almost everywhere. By Egorov's Theorem there exists for every $m \in \mathbb{N}$ a measurable subset $K_m \subset \{E \leq E_0\}$ with the property that

$$\text{vol}(\{E \leq E_0\} \setminus K_m) < \frac{1}{m} \text{ and } \lim_{n \rightarrow \infty} \sigma_n g_n = 0 \text{ uniformly on } K_m;$$

note that the set $\{E \leq E_0\}$ has finite measure. In addition we can assume that $K_m \subset K_{m+1}$, $m \in \mathbb{N}$. On the set $\{E \leq E_0\}$,

$$[\Phi_0(f_n) - \Phi_0(f_0) + (E - E_0)\sigma_n g_n] = \frac{1}{2}\Phi_0''(f_0)(\sigma_n g_n)^2 + \frac{1}{6}\Phi_0'''(f_0 + \tau\sigma_n g_n)(\sigma_n g_n)^3$$

for some $\tau \in [0, 1]$. Since both f_0 and $f_0 + \sigma_n g_n = f_n$ are non-negative the same is true for $f_0 + \tau\sigma_n g_n$, and we can use the estimate

$$|\Phi_0'''(f)| = \frac{1}{(1+f)^2} \leq 1, \quad f \geq 0$$

the estimate (3.8), and the fact that the integrand [...] in (3.8) is non-negative by (2.3) to conclude that

$$\begin{aligned} \iint_{K_m} \frac{1}{2}\Phi_0''(f_0)|g_n|^2 dv dx &= \frac{1}{\sigma_n^2} \iint_{K_m} [\dots] dv dx \\ &\quad - \frac{1}{\sigma_n^2} \iint_{K_m} \frac{1}{6}\Phi_0'''(f_0 + \tau\sigma_n g_n)(\sigma_n g_n)^3 dv dx \\ &< 1 + \frac{1}{n} + \sup_{K_m} |\sigma_n g_n| \iint |g_n|^2 dv dx. \end{aligned}$$

Taking the limit $n \rightarrow \infty$ in this estimate we find that

$$\iint_{K_m} \frac{1}{2}\Phi_0''(f_0)g^2 dv dx \leq 1.$$

If we observe the choice of the sets K_m , let $m \rightarrow \infty$, and recall the fact that $g = 0$ outside the set $\{E \leq E_0\}$ the proof of (3.2) is complete.

Proof of (3.3).

To prove (3.3), the measure-preserving property of the set \mathcal{S}_{f_0} plays the crucial role. Let $\Phi = \Phi(f, L)$ be a function as specified in the definition of that set respectively in the lemma. By Taylor expansion with respect to the first argument,

$$\Phi(f_n, L) - \Phi(f_0, L) = \partial_f \Phi(f_0, L) \sigma_n g_n + \frac{1}{2} \partial_f^2 \Phi(f_0 + \tau \sigma_n g_n, L) (\sigma_n g_n)^2$$

for some $\tau \in [0, 1]$. If we integrate this identity and observe that, since $f_n \in \mathcal{S}_{f_0}$,

$$\iint \Phi(f_n, L) dv dx = \iint \Phi(f_0, L) dv dx,$$

it follows that

$$\iint \partial_f \Phi(f_0, L) g_n dv dx = -\frac{1}{2} \sigma_n \iint \partial_f^2 \Phi(f_0 + \tau \sigma_n g_n, L) g_n^2 dv dx \rightarrow 0.$$

As to the latter limit we note that $\partial_f^2 \Phi$ is bounded, (g_n) is bounded in $L^2(\mathbb{R}^6)$, and $\sigma_n \rightarrow 0$. On the other hand $\partial_f \Phi(f_0, L)$ is supported on the compact set $\{E \leq E_0\}$ and hence bounded. Since $g_n \rightharpoonup g$ weakly in $L^2(\mathbb{R}^6)$, the identity (3.3) follows as $n \rightarrow \infty$.

Conclusion of the proof of Lemma 3.1.

The function g constructed above has all the properties required in the lemma, except that it need not be even in v . Hence we decompose it into its even and odd parts with respect to v : $g = g_{\text{even}} + g_{\text{odd}}$. We claim that (3.1), (3.2), and (3.3) remain valid for the even part. Since

$$\rho_g = \int g dv = \int g_{\text{even}} dv = \rho_{g_{\text{even}}}$$

we have $\nabla U_{g_{\text{even}}} = \nabla U_g$, and (3.1) remains valid. Since $\Phi_0''(f_0)$ is even in v ,

$$\begin{aligned} 1 &\geq \iint \frac{1}{2} \Phi_0''(f_0) (g_{\text{even}} + g_{\text{odd}})^2 dv dx = \iint \frac{1}{2} \Phi_0''(f_0) ((g_{\text{even}})^2 + (g_{\text{odd}})^2) dv dx \\ &\geq \iint \frac{1}{2} \Phi_0''(f_0) (g_{\text{even}})^2 dv dx, \end{aligned}$$

i.e., (3.2) remains valid. Finally, let Φ be as in the definition of the set \mathcal{S}_{f_0} . Then $\partial_f \Phi(f_0, L)$ is even in v so that in (3.3) the odd part of g drops out, and the assertions of Lemma 3.1 hold for g_{even} . \square

3.2 Characteristics and $g = \{f_0, h\}$

In this subsection we construct a spherically symmetric function h such that $g = \{f_0, h\}$. To this end we need to introduce variables which are adapted to the spherical symmetry:

$$r := |x|, \quad w := \frac{x \cdot v}{r}, \quad L := |x \times v|^2;$$

w is the radial velocity, and L has already been used above. Any spherically symmetric function of x and v such as the desired h can be written in terms of these variables, so $h = h(r, w, L)$. Then

$$\{f_0, h\} = \phi'_0(E)\{E, h\} = -\phi'_0(E) \left[w \partial_r + \left(\frac{L}{r^3} - U'_0(r) \right) \partial_w \right] h,$$

and the equation we wish to solve for h reads

$$\left[w \partial_r + \left(\frac{L}{r^3} - U'_0(r) \right) \partial_w \right] h = -\frac{1}{\phi'_0(E)} g. \quad (3.11)$$

In order to analyze its characteristic system

$$\dot{r} = w, \quad \dot{w} = \frac{L}{r^3} - U'_0(r)$$

we define for fixed $L > 0$ the effective potential

$$\Psi_L(r) := U_0(r) + \frac{L}{2r^2}$$

and observe that in terms of the variables r, w, L the conserved particle energy takes the form

$$E = E(x, v) = E(r, w, L) = \frac{1}{2}w^2 + \Psi_L(r);$$

L only plays the role of a parameter here since $\dot{L} = 0$. We need to analyze the effective potential Ψ_L . The boundary condition for U_0 at infinity implies that $\lim_{r \rightarrow \infty} \Psi_L(r) = 0$, and clearly, $\lim_{r \rightarrow 0} \Psi_L(r) = \infty$. Moreover,

$$U'_0(r) = \frac{4\pi}{r^2} \int_0^r \rho_0(s) s^2 ds =: \frac{m_0(r)}{r^2} > 0, \quad r > 0,$$

since U_0 is increasing, thus $U_0(0) < E_0$, and by (2.1), $\rho_0(0) > 0$. Now

$$\Psi'_L(r) = 0 \Leftrightarrow \frac{m_0(r)}{r^2} - \frac{L}{r^3} = 0 \Leftrightarrow m_0(r) - \frac{L}{r} = 0,$$

and since the left hand side of the latter equation is strictly increasing for $L > 0$ with $\lim_{r \rightarrow \infty} (m_0(r) - L/r) = M > 0$ and $\lim_{r \rightarrow 0} (m_0(r) - L/r) = -\infty$ there exists a unique $r_L > 0$ such that

$$\Psi'_L(r_L) = 0, \quad \Psi'_L(r) < 0 \text{ for } r < r_L, \quad \Psi'_L(r) > 0 \text{ for } r > r_L.$$

Moreover, since

$$\frac{d}{dr} \left(m_0(r) - \frac{L}{r} \right) = 4\pi r^2 \rho_0(r) + \frac{L}{r^2} > 0$$

the implicit function theorem implies that the mapping $]0, \infty[\ni L \mapsto r_L$ is continuously differentiable. For $r = r_L$,

$$\Psi''_L(r_L) = -2 \frac{m_0(r)}{r^3} + 3 \frac{L}{r^4} + 4\pi \rho_0(r) = 4\pi \rho_0(r) + \frac{L}{r^4} > 0. \quad (3.12)$$

The behavior of Ψ_L implies that for any $L > 0$ and $\Psi_L(r_L) < E < 0$ there exist two unique radii $0 < r_-(E, L) < r_L < r_+(E, L) < \infty$ such that

$$\Psi_L(r_{\pm}(E, L)) = E, \text{ and } \Psi_L(r) < E \Leftrightarrow r_-(E, L) < r < r_+(E, L).$$

Since $\Psi'_L(r) \neq 0$ for $r \neq r_L$ it follows again by the implicit function theorem that the mapping $(E, L) \mapsto r_{\pm}(E, L)$ is C^1 on the set $\{(E, L) \in \mathbb{R} \times]0, \infty[\mid \Psi_L(r_L) < E < 0\}$.

Let $\tau \mapsto (r(\tau), w(\tau), L)$ be a characteristic curve with $L > 0$ and running in the set $\{E \leq E_0\}$, so that $\Psi_L(r_L) < E \leq E_0 < 0$. Then

$$r(\tau) \in [r_-(E, L), r_+(E, L)], \quad w(\tau) = \pm \sqrt{2E - 2\Psi_L(r(\tau))}.$$

By the equation (3.11) which we want to solve,

$$\frac{d}{d\tau} h(r(\tau), w(\tau), L) = -\frac{1}{\phi'_0(E)} g(r(\tau), w(\tau), L),$$

which we rewrite in terms of the parameter r as

$$\frac{d}{dr} h(r, w(r), L) = -\frac{1}{\phi'_0(E)w(r)} g(r, w(r), L), \quad w(r) = \pm \sqrt{2E - 2\Psi_L(r)}.$$

Hence for $(r, w, L) \in \{E \leq E_0\}$ with $L > 0$ we define h as follows: We let $E := \frac{1}{2}w^2 + \Psi_L(r)$ so that $r_-(E, L) \leq r \leq r_+(E, L)$, and

$$h(r, w, L) := -\text{sign } w \frac{1}{\phi_0'(E)} \int_{r_-(E, L)}^r \frac{g(s, \sqrt{2E - 2\Psi_L(s)}, L)}{\sqrt{2E - 2\Psi_L(s)}} ds. \quad (3.13)$$

Outside the set $\{E \leq E_0\}$ we let $h = 0$. We need to make sure that we can consistently set $h(r, 0, L) = 0$, i.e., the integral in the above definition must vanish for $r = r_+(E, L)$. To this end let Φ be as specified in the definition of the class \mathcal{S}_{f_0} . We want to apply the change of variables $(x, v) \mapsto (r, w, L) \mapsto (r, E, L)$ to the integral in (3.3). Now

$$dx dv = 8\pi^2 dr dw dL = 8\pi^2 \frac{dr dE dL}{\sqrt{2E - 2\Psi_L(r)}}, \quad (3.14)$$

where we note that g is even in v and hence in w so that we can restrict the integral to $w > 0$. We obtain the identity

$$\begin{aligned} 0 &= \iint \partial_f \Phi(f_0, L) g dv dx = 8\pi^2 \int_0^\infty \int_0^\infty \int_0^\infty \partial_f \Phi(f_0, L) g dr dw dL \\ &= 8\pi^2 \iint_M \int_{r_-(E, L)}^{r_+(E, L)} \frac{g(r, \sqrt{2E - 2\Psi_L(r)}, L)}{\sqrt{2E - 2\Psi_L(r)}} dr \partial_f \Phi(\phi_0(E), L) dE dL, \end{aligned}$$

where $M := \{(E, L)(x, v) | f_0(x, v) > 0\}$. The class of test functions $\partial_f \Phi(\phi_0(E), L)$ is sufficiently large to conclude that for almost all E and L ,

$$\int_{r_-(E, L)}^{r_+(E, L)} \frac{g(r, \sqrt{2E - 2\Psi_L(r)}, L)}{\sqrt{2E - 2\Psi_L(r)}} dr = 0$$

as desired.

3.3 Contradiction to Lemma 1.1

As defined above, h need not be smooth or even integrable, so in order to derive a contradiction to Lemma 1.1 we need to regularize it. In order to do so it should first be noted that h as defined in (3.13) is measurable, which follows by Fubini's Theorem and the fact that by the change of variables formula the function

$$(s, r, E, L) \mapsto \frac{g(s, \sqrt{2E - 2\Psi_L(s)}, L)}{\sqrt{2E - 2\Psi_L(s)}} \mathbf{1}_{[\Psi_L(r_L), E_0]}(E) \mathbf{1}_{[r_-(E, L), r_+(E, L)]}(s) \mathbf{1}_{[0, r]}(s)$$

is integrable; $r \leq \max\{|x| \mid (x, v) \in \text{supp } f_0\}$.

The cut-off h .

As a first step in regularizing h we define for m large the set

$$\Omega_m := \left\{ (x, v) \in \mathbb{R}^6 \mid E \leq E_0 - \frac{1}{m}, L \geq \frac{1}{m} \right\}.$$

We want to approximate h by $h\mathbf{1}_{\Omega_m}$. In order to analyze this approximation the following lemma will be useful:

Lemma 3.2 *There exists a constant $C_m > 0$ such that for $L \geq \frac{1}{m}$ and $\Psi_L(r_L) < E \leq E_0$,*

$$\int_{r_-(E,L)}^{r_+(E,L)} \frac{dr}{\sqrt{2E - 2\Psi_L(r)}} < C_m.$$

Proof. We first establish the following auxiliary estimate: For all $m \in \mathbb{N}$ there exists a constant $\eta_m > 0$ such that for all $L \geq 1/m$, $\Psi_L(r_L) < E \leq E_0$, and $r \in [r_-(E, L), r_+(E, L)]$,

$$\frac{|\Psi'_L(r)|}{\sqrt{\Psi_L(r) - \Psi_L(r_L)}} \geq \eta_m. \quad (3.15)$$

To see this let $m \in \mathbb{N}$ and L, E, r be as specified. Then

$$E_0 \geq E \geq \Psi_L(r) = U_0(r) + \frac{L}{2r^2} \geq U_0(0) + \frac{1}{2mr^2}$$

and hence $r \geq (2m(E_0 - U_0(0)))^{-1/2}$. Let $R := \max\{|x| \mid (x, v) \in \text{supp } f_0\}$, i.e., $U_0(R) = E_0$. Then

$$L \leq 2r^2(E_0 - U_0(r)) \leq 2R^2(E_0 - U_0(0)).$$

Hence if we assume that (3.15) were false, we can find a sequence $(r_n, L_n) \rightarrow (\bar{r}, \bar{L})$ in the set $[(2m(E_0 - U_0(0)))^{-1/2}, R] \times [1/m, 2R^2(E_0 - U_0(0))]$ such that

$$\lim_{n \rightarrow \infty} \frac{\Psi'_{L_n}(r_n)}{\sqrt{\Psi_{L_n}(r_n) - \Psi_{L_n}(r_{L_n})}} = 0.$$

If $\bar{r} \neq r_{\bar{L}}$ it follows that $\Psi_{\bar{L}}(\bar{r}) > \Psi_{\bar{L}}(r_{\bar{L}})$ and $\Psi'_{\bar{L}}(\bar{r}) = 0$ which is a contradiction to the uniqueness of the minimizer $r_{\bar{L}}$ of $\Psi_{\bar{L}}$. So assume that $\bar{r} = r_{\bar{L}}$. Now

recall that $\Psi'_{L_n}(r_{L_n})=0$. By Taylor expansion at $r=r_{L_n}$ we find intermediate values θ_n and τ_n between r_n and r_{L_n} such that

$$\frac{|\Psi'_{L_n}(r_n)|}{\sqrt{\Psi_{L_n}(r_n)-\Psi_{L_n}(r_{L_n})}} = \frac{|\Psi''_{L_n}(\theta_n)(r_n-r_{L_n})|}{\sqrt{\frac{1}{2}\Psi''_{L_n}(\tau_n)(r_n-r_{L_n})^2}} \rightarrow \sqrt{2|\Psi''_{\bar{L}}(r_{\bar{L}})|} \neq 0, \quad n \rightarrow \infty,$$

where we recall (3.12). This contradiction completes the proof of (3.15).

To complete the proof of the lemma we split the integral as

$$\int_{r_-(E,L)}^{r_+(E,L)} \frac{dr}{\sqrt{2E-2\Psi_L(r)}} = \int_{r_-(E,L)}^{r_L} \dots + \int_{r_L}^{r_+(E,L)} \dots =: I_1 + I_2.$$

In the first term, we make a change of variables $u = \sqrt{\Psi_L(r) - \Psi_L(r_L)}$ so that $\frac{du}{dr} = \frac{1}{2u}\Psi'_L(r) < 0$ on $[r_-(E,L), r_L[$. By (3.15),

$$\begin{aligned} I_1 &= \int_0^0 \frac{1}{\sqrt{E-\Psi_L(r_L)}\sqrt{2(E-\Psi_L(r_L)-u^2)}} \frac{dr}{du} du \\ &\leq \frac{\sqrt{2}}{\eta_m} \int_0^{\sqrt{E-\Psi_L(r_L)}} \frac{du}{\sqrt{E-\Psi_L(r_L)-u^2}} = \frac{\sqrt{2}}{\eta_m} \int_0^1 \frac{ds}{\sqrt{1-s^2}} < \infty \end{aligned}$$

by a further change of variables $u = \sqrt{E-\Psi_L(r_L)}s$. The same type of estimate holds for the second part I_2 of the integral under investigation, and the proof of the lemma is complete. \square

We now show that the cut-off function $h\mathbf{1}_{\Omega_m}$ is square integrable and solves the equation $\{f_0, h\mathbf{1}_{\Omega_m}\} = g\mathbf{1}_{\Omega_m}$ in the sense of distributions, more precisely:

Lemma 3.3 *For any $m \in \mathbb{N}$ large, $h\mathbf{1}_{\Omega_m} \in L^2(\mathbb{R}^6)$, and for any spherically symmetric test function $\psi = \psi(r, w, L) \in C^1([0, \infty[\times \mathbb{R} \times [0, \infty[)$,*

$$\int_{E \leq E_0} \{f_0, \psi\} h\mathbf{1}_{\Omega_m} = - \int_{E \leq E_0} g\mathbf{1}_{\Omega_m} \psi.$$

Proof. We first prove that $h\mathbf{1}_{\Omega_m} \in L^2(\mathbb{R}^6)$. Since the integrand is even in v we can apply the change of variables (3.14):

$$\begin{aligned} &\iint \mathbf{1}_{\Omega_m} h^2 dv dx \\ &= 8\pi^2 \iint_{S_m} \int_{r_-(E,L)}^{r_+(E,L)} h^2(r, \sqrt{2E-2\Psi_L(r)}, L) \frac{dr}{\sqrt{2E-2\Psi_L(r)}} dE dL, \end{aligned}$$

where

$$S_m := \left\{ (E, L) = (E, L)(x, v) \mid (x, v) \in \text{supp } f_0, E \leq E_0 - \frac{1}{m}, L \geq \frac{1}{m} \right\}. \quad (3.16)$$

Let $(E, L) \in S_m$. In the estimates below we write $w(r) = \sqrt{2E - 2\Psi_L(r)}$ and $r_{\pm}(E, L) = r_{\pm}$ for brevity. Then by the definition (3.13) of h and the fact that $1/|\phi'_0(E)| \leq 1$ for $E \leq E_0$,

$$\begin{aligned} \int_{r_-}^{r_+} h^2(r, w(r), L) \frac{dr}{w(r)} &= \int_{r_-}^{r_+} \left[\frac{1}{\phi'_0(E)} \int_{r_-}^r g(s, w(s), L) \frac{ds}{w(s)} \right]^2 \frac{dr}{w(r)} \\ &\leq \int_{r_-}^{r_+} \left[\int_{r_-}^{r_+} g(s, w(s), L) \frac{ds}{w(s)} \right]^2 \frac{dr}{w(r)} \\ &\leq \left(\int_{r_-}^{r_+} \frac{dr}{w(r)} \right)^2 \int_{r_-}^{r_+} g^2(r, w(r), L) \frac{dr}{w(r)} \\ &\leq C_m^2 \int_{r_-}^{r_+} g^2(r, w(r), L) \frac{dr}{w(r)}; \end{aligned}$$

in the last two estimates we used the Cauchy-Schwarz inequality and Lemma 3.2. A further integration with respect to E and L and the change of variables $(r, E, L) \mapsto (x, v)$ shows that $\|h\mathbf{1}_{\Omega_m}\|_2$ is bounded in terms of C_m and $\|g\|_2$.

Now let $\psi = \psi(r, w, L)$ be a test function as specified in the lemma. Along characteristic curves of (3.11), which as before we parameterize by r distinguishing between $w > 0$ and $w < 0$, a simple computation shows that

$$\{E, \psi\} = -\text{sign } w \sqrt{2E - 2\Psi_L(r)} \frac{d}{dr} [\psi(r, \text{sign } w \sqrt{2E - 2\Psi_L(r)}, L)].$$

By the change of variables used repeatedly above, using the abbreviation

$w(r) := \sqrt{2E - 2\Psi_L(r)}$, and recalling the definition (3.16) we find that

$$\begin{aligned}
& \iint \{f_0, \psi\} h \mathbf{1}_{\Omega_m} dv dx = \int_{\{w>0\}} \dots + \int_{\{w<0\}} \dots \\
& = - \iint_{S_m} \phi'_0(E) \int_{r_-}^{r_+} w(r) \frac{d}{dr} [\psi(r, w(r), L)] h(r, w(r), L) \frac{dr}{w(r)} dE dL \\
& \quad - \iint_{S_m} \phi'_0(E) \int_{r_-}^{r_+} (-w(r)) \frac{d}{dr} [\psi(r, -w(r), L)] h(r, -w(r), L) \frac{dr}{w(r)} dE dL \\
& = \iint_{S_m} \phi'_0(E) \int_{r_-}^{r_+} \psi(r, w(r), L) \left(-\frac{g(r, w(r), L)}{\phi'_0(E) w(r)} \right) dr dE dL \\
& \quad - \iint_{S_m} \phi'_0(E) \int_{r_-}^{r_+} \psi(r, -w(r), L) \frac{g(r, -w(r), L)}{\phi'_0(E) w(r)} dr dE dL \\
& = - \iint \psi g \mathbf{1}_{\Omega_m} dv dx;
\end{aligned}$$

notice that $h(r, \pm w(r), L) = 0$ for $r = r_{\pm}(E, L)$, which together with the definition (3.13) of h along the characteristics was essential in the integration by parts above, and also that g is even in v respectively w . The proof of Lemma 3.3 is now complete. \square

Regularization of $h \mathbf{1}_{\Omega_m}$.

The function $h \mathbf{1}_{\Omega_m}$ is not smooth, hence Lemma 1.1 cannot be applied to it, and therefore we smooth it. For fixed $m \in \mathbb{N}$ the function $h \mathbf{1}_{\Omega_m}$ is, as a function of r, w, L , supported in a set of the form

$$Q_m := [R_0, R_1] \times [-W_0, W_0] \times [L_0, L_1] \quad (3.17)$$

with $0 < R_0 < R_1$, $W_0 > 0$, and $0 < L_0 < L_1$; it will be important that its support stays away both from $r = 0$ and $L = 0$. Let $\zeta \in C_c^\infty(\mathbb{R}^3)$ be even in all three variables, i.e., $\zeta(z_1, z_2, z_3) = \zeta(|z_1|, |z_2|, |z_3|)$, $\zeta \geq 0$, $\int \zeta = 1$, and define $\zeta_n := n^3 \zeta(n \cdot)$ for $n \in \mathbb{N}$. For n sufficiently large we define

$$h_n(r, w, L) = \int_0^\infty \int_{-\infty}^\infty \int_0^\infty (h \mathbf{1}_{\Omega_m})(\bar{r}, \bar{w}, \bar{L}) \zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L}) d\bar{L} d\bar{w} d\bar{r}.$$

Since Ω_m is strictly inside the set $\{E \leq E_0\}$ with a positive distance from its boundary, $h_n \in C_c^\infty([0, \infty[\times \mathbb{R} \times]0, \infty[)$, and without loss of generality we can assume that $\text{supp } h_n \subset Q_m \cap \{E < E_0\}$. Since h is odd in w respectively v so is h_n , and clearly, $h_n \rightarrow h \mathbf{1}_{\Omega_m}$ in $L^2 \cap L^1$. The crucial step is to show that

$$\lim_{n \rightarrow \infty} \{f_0, h_n\} = g \mathbf{1}_{\Omega_m} \quad (3.18)$$

in L^2 . To this end we fix (r, w, L) , write $\bar{E} := E(\bar{r}, \bar{w}, \bar{L})$ and $\int := \int_0^\infty \int_{-\infty}^\infty \int_0^\infty$ for brevity, and split the convolution integral into three parts:

$$\begin{aligned}
\{f_0, h_n\} &= -\phi'_0(E) \left[w \partial_r + \left(\frac{L}{r^3} - U'_0(r) \right) \partial_w \right] h_n \\
&= - \int (h \mathbf{1}_{\Omega_m})(\bar{r}, \bar{w}, \bar{L}) (\phi'_0(E) - \phi'_0(\bar{E})) \left[w \partial_r + \left(\frac{L}{r^3} - U'_0(r) \right) \partial_w \right] \\
&\quad \zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L}) d\bar{L} d\bar{w} d\bar{r} \\
&+ \int (h \mathbf{1}_{\Omega_m})(\bar{r}, \bar{w}, \bar{L}) \phi'_0(\bar{E}) \left[\bar{w} \partial_{\bar{r}} + \left(\frac{\bar{L}}{\bar{r}^3} - U'_0(\bar{r}) \right) \partial_{\bar{w}} \right] \\
&\quad \zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L}) d\bar{L} d\bar{w} d\bar{r} \\
&- \int (h \mathbf{1}_{\Omega_m})(\bar{r}, \bar{w}, \bar{L}) \phi'_0(\bar{E}) \left[(\bar{w} - w) \partial_{\bar{r}} + \left(\frac{\bar{L}}{\bar{r}^3} - \frac{L}{r^3} - U'_0(\bar{r}) + U'_0(r) \right) \partial_{\bar{w}} \right] \\
&\quad \zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L}) d\bar{L} d\bar{w} d\bar{r} \\
&=: -I_1 + I_2 - I_3.
\end{aligned}$$

By Lemma 3.3,

$$I_2 = \int g \mathbf{1}_{\Omega_m}(\bar{r}, \bar{w}, \bar{L}) \zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L}) d\bar{L} d\bar{w} d\bar{r} \rightarrow g \mathbf{1}_{\Omega_m} \text{ in } L^2.$$

We show that I_1 and I_3 tend to zero in L^2 . To do so we change the integration variables into $\tilde{r} = n(r - \bar{r})$, $\tilde{w} = n(w - \bar{w})$, $\tilde{L} = n(L - \bar{L})$ so that $\bar{r} = r - \tilde{r}/n$, $\bar{w} = r - \tilde{w}/n$, $\bar{L} = r - \tilde{L}/n$, $d\bar{L} d\bar{w} d\bar{r} = n^{-3} d\tilde{L} d\tilde{w} d\tilde{r}$, and $\partial_r \zeta_n(r - \bar{r}, w - \bar{w}, L - \bar{L}) = n^4 \partial_{\tilde{r}} \zeta(\tilde{r}, \tilde{w}, \tilde{L})$ with analogous formulas for the other derivatives. Thus

$$\begin{aligned}
I_1 &= \int (h \mathbf{1}_{\Omega_m})(\bar{r}, \bar{w}, \bar{L}) \frac{\phi'_0(E) - \phi'_0(\bar{E})}{1/n} \\
&\quad \left[w \partial_{\bar{r}} + \left(\frac{L}{r^3} - U'_0(r) \right) \partial_{\bar{w}} \right] \zeta(\tilde{r}, \tilde{w}, \tilde{L}) d\tilde{L} d\tilde{w} d\tilde{r}, \\
I_3 &= \int (h \mathbf{1}_{\Omega_m})(\bar{r}, \bar{w}, \bar{L}) \phi'_0(\bar{E}) \\
&\quad \left[-\tilde{w} \partial_{\tilde{r}} + \left(-\frac{\tilde{L}}{\tilde{r}^3} + L \frac{\tilde{r}^{-3} - r^{-3}}{1/n} + \frac{U'_0(\bar{r}) - U'_0(r)}{1/n} \right) \partial_{\tilde{w}} \right] \zeta(\tilde{r}, \tilde{w}, \tilde{L}) d\tilde{L} d\tilde{w} d\tilde{r}
\end{aligned}$$

Now

$$\frac{\phi'_0(E) - \phi'_0(\bar{E})}{1/n} \rightarrow -\phi''_0(E) \left(w\tilde{w} + \frac{\tilde{L}}{2r^2} - \frac{\tilde{r}L}{r^3} + \tilde{r}U'_0(r) \right),$$

$$\frac{\bar{r}^{-3} - r^{-3}}{1/n} \rightarrow \frac{3\bar{r}}{r^4}, \quad \frac{U'_0(\bar{r}) - U'_0(r)}{1/n} \rightarrow -U''_0(r)\bar{r}$$

for $n \rightarrow \infty$, and all these limits are uniform with respect to $(r, w, L) \in Q_m$ defined in (3.17) and with respect to $(\tilde{r}, \tilde{w}, \tilde{L}) \in \text{supp}\zeta$. Hence

$$\begin{aligned} I_1 &\rightarrow -(h\mathbf{1}_{\Omega_m})(r, w, L)\phi''_0(E) \\ &\int \left[w\tilde{w} + \frac{\tilde{L}}{2r^2} - \frac{\tilde{r}L}{r^3} + \tilde{r}U'_0(r) \right] \left[w\partial_{\tilde{r}} + \left(\frac{L}{r^3} - U'_0(r) \right) \partial_{\tilde{w}} \right] \zeta(\tilde{r}, \tilde{w}, \tilde{L}) d\tilde{L} d\tilde{w} d\tilde{r} \\ &= 0, \\ I_3 &\rightarrow (h\mathbf{1}_{\Omega_m})(r, w, L)\phi'_0(E) \\ &\int \left[-\tilde{w}\partial_{\tilde{r}} + \left(-\frac{\tilde{L}}{\tilde{r}^3} + \frac{3L\tilde{r}}{r^4} - U''_0(r)\tilde{r} \right) \partial_{\tilde{w}} \right] \zeta(\tilde{r}, \tilde{w}, \tilde{L}) d\tilde{L} d\tilde{w} d\tilde{r} \\ &= 0 \end{aligned}$$

where both limits are in L^2 and the zeroes result from integration by parts with respect to \tilde{r} and \tilde{w} and an exact cancellation. The assertion (3.18) is now established.

Finally: The desired contradiction.

The functions h_n have all the properties required in Lemma 1.1, and we obtain the estimate

$$D^2\mathcal{H}_C(f_0)[\{f_0, h_n\}] \geq -\frac{1}{2} \int_{\{E < E_0\}} \phi'_0(E) \frac{1}{r} U'_0(r) |h_n|^2 dv dx.$$

By (3.18) and the fact that h_n and $g\mathbf{1}_{\Omega_m}$ are supported in a common compact set, $\nabla U_{\{f_0, h_n\}} \rightarrow \nabla U_{g\mathbf{1}_{\Omega_m}}$ in L^2 as $n \rightarrow \infty$. Since $\frac{1}{r}U'_0(r)$ is bounded on $[0, \infty[$ we obtain the estimate

$$D^2\mathcal{H}_C(f_0)[g\mathbf{1}_{\Omega_m}] \geq -\frac{1}{2} \int_{\{E < E_0\}} \phi'_0(E) \frac{1}{r} U'_0(r) h^2 \mathbf{1}_{\Omega_m} dv dx.$$

Since $\lim_{m \rightarrow \infty} g\mathbf{1}_{\Omega_m} = g$ in $L^1 \cap L^2(\mathbb{R}^6)$, there exists m_0 sufficiently large such that $g\mathbf{1}_{\Omega_{m_0}} \neq 0$ and by Lemma 3.3, $h\mathbf{1}_{\Omega_{m_0}} \neq 0$. For all $m \geq m_0$, $\Omega_{m_0} \subset \Omega_m$ and $h^2 \mathbf{1}_{\Omega_{m_0}} \leq h^2 \mathbf{1}_{\Omega_m}$. Since $\phi'_0(E) < 0$ and $\frac{1}{r}U'_0(r) > 0$ we have for all $m \geq m_0$,

$$D^2\mathcal{H}_C(f_0)[g\mathbf{1}_{\Omega_m}] \geq -\frac{1}{2} \int_{\{E < E_0\}} \phi'_0(E) \frac{1}{r} U'_0(r) h^2 \mathbf{1}_{\Omega_{m_0}} dv dx =: C > 0.$$

Hence $m \rightarrow \infty$ leads to a contradiction to Lemma 3.1, and the proof of Theorem 2.2 is complete.

4 Appendix: Proof of Lemma 1.1

Let

$$U_h(x) := \iint \{f_0, h\} dv \frac{dy}{|x-y|}$$

be the potential induced by $-\{f_0, h\}$, which is spherically symmetric. A short computation using the definition (1.6) of the Lie-Poisson bracket shows that

$$\int \{f_0, h\} dv = \nabla_x \cdot \int v \phi'_0(E) h(x, v) dv.$$

By the spherical symmetry of U_h and h ,

$$\begin{aligned} U'_h(r) &= \frac{1}{r^2} \int_{|x| \leq r} \nabla_x \cdot \int v \phi'_0(E) h(x, v) dv \\ &= \frac{1}{r^2} \int_{|x|=r} \int \phi'_0(E) h(x, v) v \cdot \frac{x}{r} dv d\omega(x) = 4\pi \int w \phi'_0(E) h(x, v) dv. \end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality,

$$\frac{1}{8\pi} \int |\nabla_x U_h|^2 dx \leq 2\pi \int \left[- \int w^2 \phi'_0(E) dv \right] \left[- \int \phi'_0(E) h^2 dv \right] dx.$$

Since

$$w^2 \phi'_0(E) = w \frac{d}{dw} \phi_0 \left(\frac{1}{2} w^2 + \frac{L}{2r^2} + U_0(r) \right)$$

an integration by parts with respect to w yields

$$- \int w^2 \phi'_0(E) dv = \frac{\pi}{r^2} \int_0^\infty \int_{-\infty}^\infty \phi_0 \left(\frac{1}{2} w^2 + \frac{L}{2r^2} + U_0(r) \right) dw dL = \rho_0(r).$$

Hence

$$D^2 \mathcal{H}_C(f_0)[\{f_0, h\}] \geq -\frac{1}{2} \iint \phi'_0(E) (|\{E, h\}|^2 - 4\pi \rho_0 h^2) dv dx.$$

Since h is odd in v respectively w the function

$$\mu(r, w, L) := \frac{1}{rw} h(r, w, L)$$

is smooth away from $r=0$; in passing we notice that the functions h_n to which we applied Lemma 1.1 in the proof of Theorem 2.2 have support bounded away from $r=0$, but this is not necessary for Lemma 1.1. Since $h=rw\mu$,

$$\{E, h\} = rw\{E, \mu\} + \mu\{E, rw\},$$

and hence

$$\begin{aligned} |\{E, h\}|^2 &= (rw)^2|\{E, \mu\}|^2 + rw\{E, rw\}\{E, \mu^2\} + \mu^2|\{E, rw\}|^2 \\ &= (rw)^2|\{E, \mu\}|^2 + \{E, \mu^2rw\{E, rw\}\} - \mu^2rw\{E, \{E, rw\}\}. \end{aligned}$$

The first term is as claimed in Lemma 1.1. The second term leads to $\{f_0, \mu^2rw\{E, rw\}\}$ whose integral with respect to x and v vanishes; if we cut a small ball of radius ϵ about $x=0$ from the x -integral then the surface integral appearing after the integration by parts with respect to x vanishes for $\epsilon \rightarrow 0$ since $r\mu^2 \leq C/r$. By the Poisson equation,

$$\{E, \{E, rw\}\} = -2wU'_0 - w(rU'_0)' = -rw \left(4\pi\rho_0 + \frac{1}{r}U'_0 \right).$$

Hence the third term above becomes $4\pi\rho_0h^2 + h^2\frac{1}{r}U'_0$, and Lemma 1.1 is proven. \square

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