For the volume of the fundamental parallelotopes this gives

$$\mathrm{Vol}_{F_B} = \sqrt{\det(G(B))} = \sqrt{\det(G(A)) \cdot \det(M)^2} = \mathrm{Vol}_{F_A}\,.$$

$\square$

Let $b^{(1)}, b^{(2)}, \ldots, b^{(m)}$ be a basis of a lattice $L \subset \mathbb{R}^n$ of rank m. From the above proof we see that for each unimodular matrix $M \in \mathbb{Z}^{m\times m}$ the columns of the matrix $B \cdot M$ form another basis of L, i. e. there is a one-to-one correspondence between the unimodular matrices $\in \mathbb{Z}^{m\times m}$ and the bases of L.

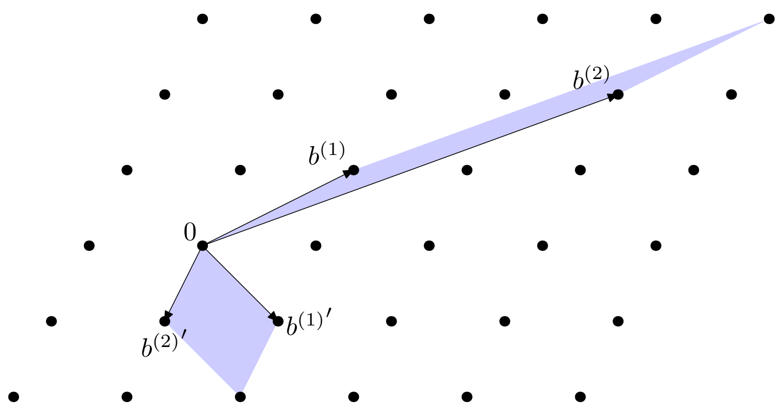

Figure 1.2: Two different bases for $b^{(1)}, b^{(2)}$ and $b^{(1)\prime}, b^{(2)\prime}$ of the same lattice.

The fundamental parallelotope is one example of a fundamental region. Every space tiling with just one lattice point in each copy is called *fundamental region*. The volumes of all fundamental regions in a lattice L are equal. Other invariants of a lattice which are independent from the choice of the basis are the *successive minima* introduced by Minkowski [118].

Definition 1.4 For a lattice $L \subset \mathbb{R}^n$ of rank m the *i-th minimum* $\lambda_i(L)$ $(1 \le i \le m)$ is defined as the least positive real number t, for which there exist i linear independent lattice vectors $v \in L \setminus \{0\}$ with $\|v\|_2^2 \le t$.

Clearly,

$$\lambda_1(L) \le \lambda_2(L) \le \cdots \le \lambda_m(L)\,.$$

t-(v, k, λ) designs for arbitrary values of λ. Therefore, the order of the prescribed group of automorphisms has to be chosen much smaller, which results in much larger matrices in (3.3). Then, the LLL-reduction phase of Algorithm 1.27—albeit running in polynomial time—takes too much time compared to a naive backtracking approach as described in many sources [59, 83, 109, 110].

Multidimensional subset sum problems of the form $A \cdot x = J$, x a $\{0, 1\}$-vector, and J equal to the all-one vector $(1, 1, \ldots, 1)^\top$, are also called *exact cover problems*. Knuth [83] gives details how to implement the backtracking algorithm, as described in [59], efficiently.

In [164] a parallel version of the algorithm of Knuth is described. The parallelization strategy there is similar to the strategy in Section 1.5. It enabled the complete solution of the Aztec diamond challenge, a problem posed by Knuth [83, 164], compare Fig. 3.2.

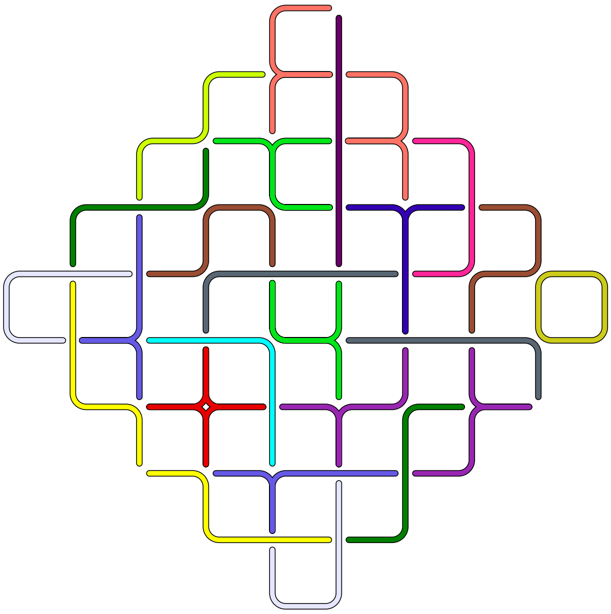

Figure 3.2: One of 107 nonisomorphic solutions of the Aztec diamond challenge.

With different implementations of the algorithm formally described in [59]